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# Edwin Soeryadjaya Problem
## Theoretical 3: Mirage

The refractive index of the air varies with temperature. Cold air is denser than warm air and has therefore a greater refractive index. Thus a temperature gradient in the atmosphere is always associated with a gradient of the refractive index. Under certain conditions, this gradient of the refractive index could be strong enough so that light rays are bent to produce a displaced image of distant objects. This amazing phenomenon is called a mirage. Mirages can be categorized as "inferior" and "superior". Inferior mirages can be seen on deserts and highways, and superior mirages occur over the sea.

To describe in detail the phenomenon of the mirage, we need to analyze the path of light rays in media with a refractive index gradient.

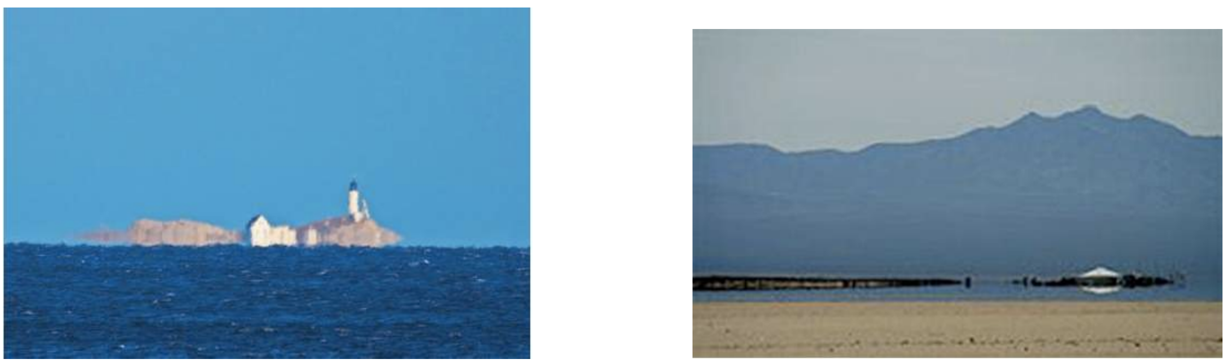

Figure 1: Left: A superior image. Right: An inferior image

## 1 The path of a light ray and the trajectory of a mass point

Media with a refractive index gradient can be treated as being formed from thin homogeneous layers with different refractive indices. The path of a light ray can be determined then by analyzing refractions of the light ray at interfaces between these thin homogeneous layers. But there exists a more convenient way for determining the path of a light ray in media with a refractive index gradient. Instead of analyzing the propagation of the light, one may study the motion of a mass point that moves along the path of the light ray, driven by a conservative force. The potential energy of the conservative force field depends on the distribution the refractive index. Once the potential energy is established, one may study the motion of the mass point by using well developed tools in classical mechanics, and find the trajectory of a mass point which is also the path of the light ray.

(a) Refraction of a light ray at the interface between a medium with a refractive index $n_1$ and a medium with a refractive index $n_2$ is shown in Fig. 2a. The angles $i_1$ and $i_2$ obey the Snell's law

$$n_1 \sin i_1 = n_2 \sin i_2 \tag{1}$$

The path of a mass point in a conservative force field is illustrated in Fig. 2b. The speed of the mass point is $v_1$ in the region with the potential energy $E_{p1}$, and $v_2$ in the region with the potential energy $E_{p2}$. Find expressions for $v_1$ and $v_2$ so that the relation $n_1 \sin i_1 = n_2 \sin i_2$ hold also for the trajectory of the mass point. You may use an arbitrary constant expressions $v_0$ with the dimension of speed in your expressions. (0.2 points)

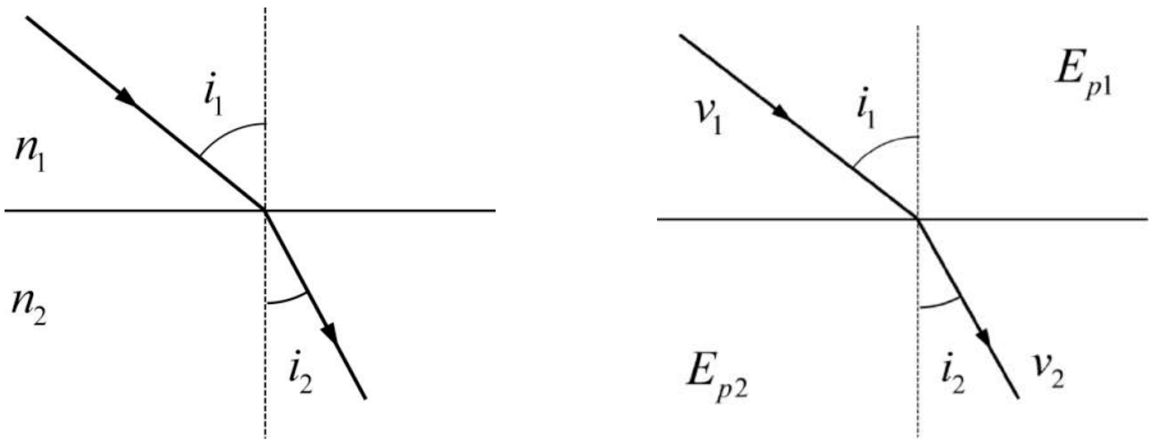


Figure 2: (a) Left: Refraction of a light ray. (b) Right: Deviation of a mass point in a conservative force field

(b) Assume that the mass of the mass point is $m$, and the total energy of the mass point equals to zero. Find the potential energies $E_{p1}$ and $E_{p2}$ in $v_0$, $n_1$ and $n_2$. (0.2 points)

(c) The trajectory of a mass point with a mass $m$ in a conservative force field is the same as the path of a light ray in a medium with a refractive index $n(\vec{r})$ which is a function of the position. The total energy of this mass point is zero. Find expressions for the potential energy $E_p(\vec{r})$ of the conservative force field and the speed of the mass point $v(\vec{r})$. (0.4 points)

(d) To describe the motion of a mass point, one expresses the position of the mass point as a function of time: $\vec{r}(t)$. To describe the trajectory of a mass point, one needs to express the position of the mass point as a function of the distance $s$ traveled by the mass point from the start point of the trajectory to the current position: $\vec{r}(s)$. Derive a differential equation for the trajectory $\vec{r}(s)$ of a mass point with a mass $m$ and a total energy $E$ that moves in a conservative force field. (0.6 points)

**Hint:** in a conservative force field, $\vec{r}(t)$ satisfies Newtons second law:

$$m\frac{d^2\vec{r}}{dt^2} = -\vec{\nabla}E_p\,, \tag{2}$$

where

$$\frac{ds}{dt} = v(\vec{r})\,, \tag{3}$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\,. \tag{4}$$

(e) Derive the light ray equation (a differential equation for the path of a light ray) by using the results obtained in (c) and (d). (0.6 points)

**Hint:** $\vec{\nabla} f^2(\vec{r}) = 2f(\vec{r})\vec{\nabla} f(\vec{r})$

## 2 The inferior mirage

When an inferior mirage appears, an image of a distant object can be seen under the real object. A direct image of that object is seen because some of the light rays enter the eye in a straight line without being refracted. The double images seem to be that of the object and its upside-down reflection in water. For exhausted travelers in the desert it seems like that there is a lake of water in front of them.

An inferior mirage occurs when a strong positive gradient of refractive index is present near the ground. We use the following model to describe the variation of the refractive index of the air with elevation:

$$n^2(z) = \begin{cases} n_0^2 + \alpha z & \text{for } h \geq z \geq 0 \\ n_0^2 + \alpha h & \text{for } z > h \end{cases} \tag{5}$$

with $n_0^2 - 1 \ll 1$ and $\alpha h \ll 1$.

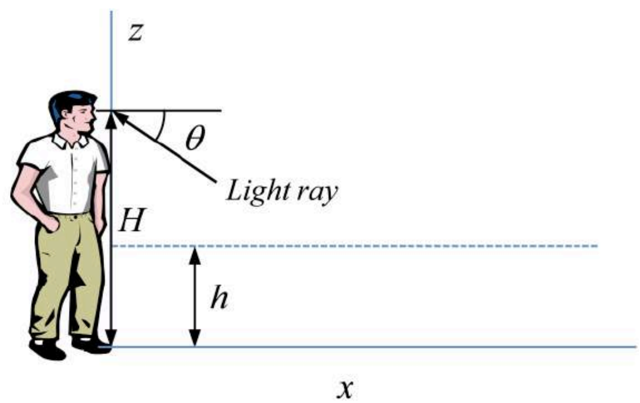


Figure 3: The geometry for analyzing an inferior mirage

(a) Find the path of a light ray (i.e. $z$ as a function of $x$ ) that enters in an observer's eye at an angle $\theta$ (see Fig. 3). The height of the observers eye is $H$. (1.9 points)

(b) Due to the inferior mirage, the observer can see an inverted image of the upper part of a camel at a large distance. The parameters for the refractive index of the air are $\alpha = 3.0 \times 10^{-5}$ m$^{-1}$, $h = 0.50$ m. The height of the observers eye is $H = 1.5$ m, and the height of the camel is $l = 2.2$ m. Find the minimum values for the distance $D_m$ between the observer and the camel so that the observer still can see the inverted image of the upper part of the camel. You may use the approximation $n_0^2 \approx 1$. (0.7 points)

(c) As a result of the light refraction in the region with refractive index gradient, the observer cannot see the lower part of the camels legs. Find the height of the lowest point ($l_m$) on the camel at the distance $D_m$ that can still be seen by the observer. (0.7 points)

(d) Find the distance $d$ between the observer and the imaginary lake of water. (0.3 points)

## 3 The superior mirage

When a superior mirage occurs, light rays that were originally directed above the line of sight of the observer will reach the observer's eyes. Thus, an object ordinarily below the horizon will be apparently above the horizon.

A superior mirage occurs when a negative gradient of refractive index is present over a body of water or over large sheets of ice. We use the following model to describe the refractive index of the atmosphere (see Fig. 4):

$$n^2(r) = \begin{cases} n_0^2\,[1-\beta(r-r_0)] & \text{for } b \geq (r-r_0) \geq 0 \\ n_0^2\,[1-\beta b] & \text{for } (r-r_0) > b \end{cases} \tag{6}$$

where $r_0$ is the radius of the earth, $b \ll r_0$ and $\beta b \ll 1$.

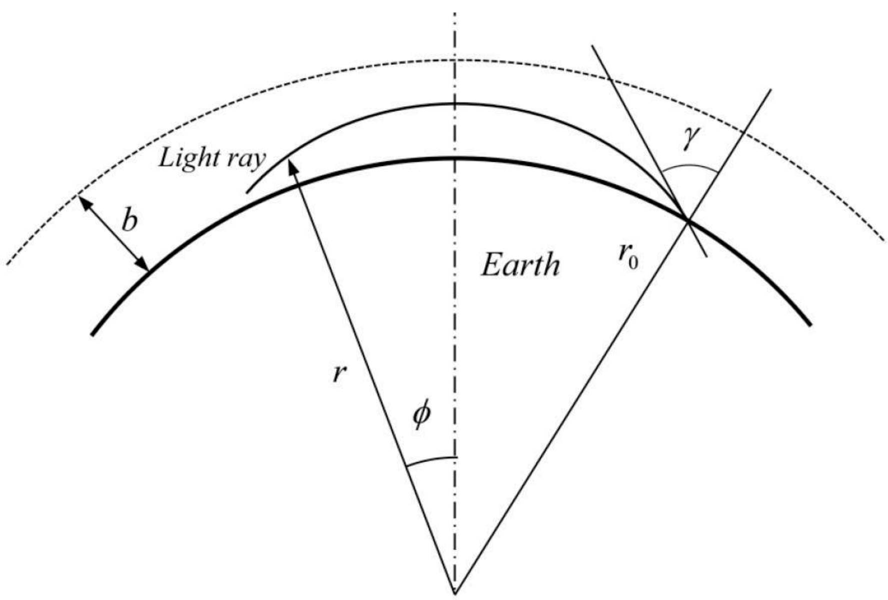


Figure 4: The geometry for analyzing a superior mirage

(a) Find the path of a light ray (i.e. $r$ as a function of $\phi$ ) within the range $b \geq (r-r_0) \geq 0$. Use $\gamma$, the angle between the light ray and the vertical direction at the sea level as the parameter of the path. (1.4 points)

**Hint:** The trajectory of a mass point with mass $m$, angular momentum $L$ and total energy $E$ in a conservative force field with

$$E_p = -\frac{mA}{r} + E_0 \tag{7}$$

is

$$r = \frac{\rho}{1 - \varepsilon\cos\phi}\,, \tag{8}$$

where

$$\rho = \frac{L^2}{m^2 A}\,, \qquad \text{and} \qquad \varepsilon = \sqrt{1 + \frac{2(E - E_0)L^2}{m^3 A^2}}\,. \tag{9}$$

We also have

$$\frac{1}{r} \approx \frac{1}{r_0} - \frac{r - r_0}{r_0^2} \qquad \text{if} \quad |r - r_0| \ll r_0. \tag{10}$$

(b) Find the minimum value of $\beta$ ($\beta_m$) at which the superior mirage occurs. Use the following values for $n_0$ and $r_0$: $n_0 \approx 1$, $r_0 = 6.4 \times 10^6$ m. (0.8 points)

(c) Under a certain atmospheric condition, with $b = 100$ m and $\beta = 6.0 \times 10^{-7}$ m$^{-1}$, calculate the largest distance $D_M$ at which the surface of the sea can be seen by an observer at an altitude $y = 10$ m ($y$ is the altitude of the observer's eye). (1.4 points)

**Useful formula:** $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ for $\phi \ll 1$.

(d) For comparison, calculate the largest distance $D'_M$ at which the surface of the sea can be seen by an observer at the same altitude $y = 10$ m, when the refractive index of the air is constant. (0.3 points)

(e) Calculate the angular difference $\Delta\vartheta$ between the apparent horizon when a superior mirage occurs as described in 3(c) and the apparent horizon in a normal day as described in 3(d), seen at the same altitude $y = 10$ m. (0.5 points)