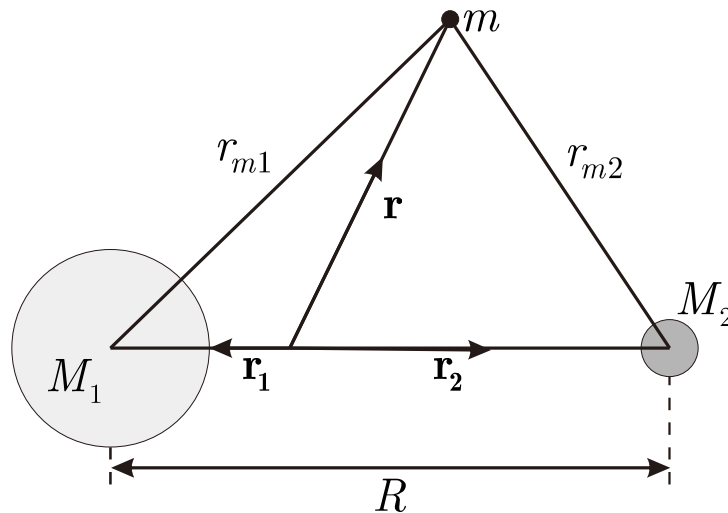


Lagrange Points Stability

In a system that rotates with the Earth around the Sun, there are five equilibrium points (where the sum of the forces is zero). These 5 points are known as Lagrange Points (named after Joseph Lagrange, the first person to study this three-body system). Exact analysis of this system is very complicated and chaotic. In the following problem, the mass of the two bodies (M_1 and M_2) are taken to be much larger than that of the third body (m). The distance between M_1 and M_2 is taken to be R .



1. Basic equations of the system

- (a) Write down the vector of the total gravitational forces \mathbf{F}_g on m .
- (b) By assuming $M_1, M_2 \gg m$, determine the angular velocity of the M_1 and M_2 system (Ω).
- (c) In a frame that rotates with the system, there are fictitious forces on m . Write down the vector of the total forces on this mass (\mathbf{F}_Ω) in this frame.
- (d) Choose a coordinate system where the three masses are in the xy -plane and the angular velocity Ω is in the positive z -axis. The center of the coordinate is set at the center of mass of M_1 and M_2 on the x -axis. Write the position of m as $\mathbf{r} = x(t)\hat{i} + y(t)\hat{j}$. In the rotating frame where m is at rest, write down the total forces on m in the x - and y -axis using parameter $\alpha = \frac{M_2}{M_1+M_2}$ and $\beta = \frac{M_1}{M_1+M_2}$.

2. Identifying Lagrange Points

There are 5 points with zero net forces in this rotating system. Three of them (call them L_1 , L_2 and L_3) lie on the line connecting M_1 and M_2 (the x -axis) and the other two (call them L_4 and L_5) lie on the xy -plane on symmetric positions above and below the x -axis; that is, $y_4 = -y_5$.

- (a) First consider the case of finding the position of L_1 , L_2 and L_3 . Use $x = (\nu - \alpha)R$, with ν the distance of m from M_1 in units of R . Write down the equation of force that must be satisfied to identify these points. Express this equation in terms of ν and α .
- (b) The equation above gives rise to three cases (each for L_1 , L_2 and L_3) to consider, $\nu < a$, $a < \nu < b$ and $b < \nu$. Determine the values of a and b .

From here on, we will also assume that α is small (in the Earth-Sun system, α is 3.0×10^{-6}). Use only the lowest order non-zero term in α , ignore all higher order terms in α . The following three questions will help you determine the three Lagrange points on the x -axis.

- (c) For the first case, $\nu < a$, write $\nu = -1 + \delta_1$ with δ_1 a small positive number that depends on α . This value of ν will determine the position of the first Lagrange point at $x = -R(1 + \xi_1)$. Determine ξ_1 as a function of α .
- (d) For the second case, $a < \nu < b$, write $\nu = 1 - \delta_2$ with δ_2 a small positive number that depends on α . This value of ν will determine the position of the second Lagrange point at $x = R(1 - \xi_2)$. Determine ξ_2 as a function of α .
- (e) For the third case, $b < \nu$, write $\nu = 1 + \delta_3$ with δ_3 a small positive number that depends on α . This value of ν will determine the position of the third Lagrange point at $x = R(1 + \xi_3)$. Determine ξ_3 as a function of α .

Determining the fourth and fifth Lagrange points requires a more complicated method. First decompose the gravitational force on m into components parallel and perpendicular to the vector \mathbf{r} .

- (f) Find the unit vector parallel to the vector \mathbf{r} , $\hat{\mathbf{e}}_{\parallel}$. Find also the unit vector perpendicular to the vector \mathbf{r} on the xy -plane, $\hat{\mathbf{e}}_{\perp}$.
- (g) Find the component of the force on m parallel to the vector \mathbf{r} , F_{Ω}^{\parallel} , and find the component perpendicular to the vector \mathbf{r} , F_{Ω}^{\perp} .
- (h) Specify the condition that must be satisfied by the force component perpendicular to the vector \mathbf{r} in order that mass m be in equilibrium. With this condition, determine the relation between r_{m1} and r_{m2} .
- (i) Specify the condition that must be satisfied by the force component parallel to the vector \mathbf{r} in order that mass m be in equilibrium. With this equation, determine the relation between r_{m1} and R .
- (j) Now determine the position of the fourth Lagrange point (x_4, y_4) and the fifth Lagrange point (x_5, y_5) .

3. Lagrange Point Stability

To test the stability of these Lagrange points, small perturbation are given to the mass m around its equilibrium points. Because the forces in this system depend on the

position (x, y) and the velocity (v_x, v_y) of the mass m , the restoring forces must be calculated for variations in position and velocity. Expand the total force as follows:

$$F_x(x_0 + \delta x, y_0 + \delta y, v_{x,0} + \delta v_x, v_{y,0} + \delta v_y) = \frac{\partial F_x}{\partial x} \delta x + \frac{\partial F_x}{\partial y} \delta y + \frac{\partial F_x}{\partial v_x} \delta v_x + \frac{\partial F_x}{\partial v_y} \delta v_y$$

$$F_y(x_0 + \delta x, y_0 + \delta y, v_{x,0} + \delta v_x, v_{y,0} + \delta v_y) = \frac{\partial F_y}{\partial x} \delta x + \frac{\partial F_y}{\partial y} \delta y + \frac{\partial F_y}{\partial v_x} \delta v_x + \frac{\partial F_y}{\partial v_y} \delta v_y.$$

This force has taken into account the contribution of the velocity of the mass m . All the partial derivatives are evaluated at the point $(x_0, y_0, v_{x,0}, v_{y,0})$.

- Write down the general form for $\frac{1}{m} \frac{\partial F_x}{\partial x}$, $\frac{1}{m} \frac{\partial F_x}{\partial y}$, $\frac{1}{m} \frac{\partial F_y}{\partial x}$, $\frac{1}{m} \frac{\partial F_y}{\partial y}$. Show that $\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$.
- Calculate $\frac{1}{m} \frac{\partial F_x}{\partial v_x}$, $\frac{1}{m} \frac{\partial F_x}{\partial v_y}$, $\frac{1}{m} \frac{\partial F_y}{\partial v_x}$, $\frac{1}{m} \frac{\partial F_y}{\partial v_y}$. These four coefficients should not change for the five Lagrange points.

These eight coefficients should act as a restoring constant (analog to the spring constant). Now we are ready to check the stability of the five Lagrange points. Consider only the lowest order term in α , ignore all higher order terms.

- The first Lagrange Point
 - Show that $\frac{1}{m} \frac{\partial F_x}{\partial x} = c_1 \Omega^2$. Determine c_1 .
 - Show that $\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y} = 0$.
 - Show that $\frac{1}{m} \frac{\partial F_y}{\partial y} = c_2 \alpha \Omega^2$. Determine c_2 .
 - By substituting $\delta x = Ae^{\lambda t}$ and $\delta y = Be^{\lambda t}$, with A and B nonzero, determine λ as a function of α and Ω only.
 - There are four solutions to λ . Write down the condition that these solutions must satisfy in order that the first Lagrange point is stable and then determine the stability of this point.
 - For the Earth-Sun system α is 3.0×10^{-6} and Ω is $2\pi/\text{year}$. If this point is stable, determine its period of oscillation (in days), if not, determine its time constant $1/\lambda$ (in days also).
- The second Lagrange Point
 - Show that $\frac{1}{m} \frac{\partial F_x}{\partial x} = c_3 \Omega^2$. Determine c_3 .
 - Show that $\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y} = 0$.
 - Show that $\frac{1}{m} \frac{\partial F_y}{\partial y} = c_4 \Omega^2$. Determine c_4 .
 - By substituting $\delta x = Ae^{\lambda t}$ and $\delta y = Be^{\lambda t}$, with A and B nonzero, determine λ as a function of α and Ω only.
 - There are four solutions to λ . Write down the condition that these solutions must satisfy in order that the second Lagrange point is stable and then determine the stability of this point.

- vi. For the Earth-Sun system: if this point is stable, determine its period of oscillation (in days), if not, determine its time constant $1/\lambda$ (in days also).

The third Lagrange point is similar to the second Lagrange point hence it need not be considered.

(e) The fourth Lagrange Point

- i. Show that $\frac{1}{m} \frac{\partial F_x}{\partial x} = c_5 \Omega^2$. Determine c_5 .
- ii. Show that $\frac{1}{m} \frac{\partial F_y}{\partial x} = \frac{1}{m} \frac{\partial F_x}{\partial y} = (c_6 + c_7 \alpha) \Omega^2$. Determine c_6 and c_7 .
- iii. Show that $\frac{1}{m} \frac{\partial F_y}{\partial y} = c_8 \Omega^2$. Determine c_8 .
- iv. By substituting $\delta x = Ae^{\lambda t}$ and $\delta y = Be^{\lambda t}$, with A and B nonzero, determine λ as a function of α and Ω only.
- v. Define $M_1/M_2 = \xi$. Find the range of value of ξ for the fourth Lagrange point to be stable.

The fifth Lagrange point has the same behavior as the fourth Lagrange point, hence it need not be considered.