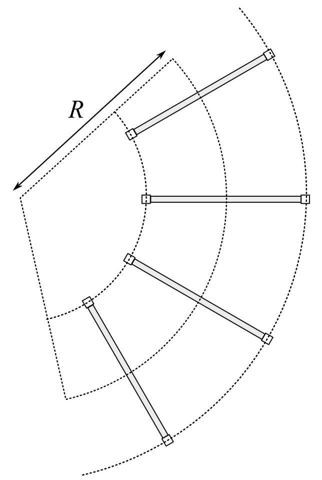
Solution: Motion of a rolling rod



A) *a)* The initial velocity and the angular velocity of the rod is

$$v_0 = \frac{v_1 + v_2}{2}, \qquad \omega_0 = \frac{|v_1 - v_2|}{L}.$$

b) Since the mass and the radius of the castor-wheels are negligibly small, the force of static friction has only a component parallel to the rod (otherwise the component of the static friction perpendicular to the rod cause an infinitely large angular acceleration.). From this, it follows that the torque acting on the rod is zero, *i.e.* the angular velocity of the rod is constant:

$$\omega(t) = \omega_0.$$

Since all the forces acting on the rod is parallel of its axis, the magnitude of the velocity of the center of the rod remains constant:

$$|\mathbf{v}(t)| = v_0,$$

and the direction of $\mathbf{v}(t)$ is always perpendicular to the rod. So the orbit of the center of mass will be a circle (see the upper Figure).
During one period ($2\pi/\omega_0$) the center of the rod draw a whole circle, so one can write:

$$\frac{2\pi}{\omega_0} v_0 = 2\pi R,$$

where R is the radius of the circle. From this, it follows that the radius of the orbit of the center of mass is

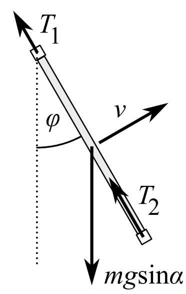
$$R = \frac{v_0}{\omega_0} = \frac{v_1 + v_2}{|v_1 - v_2|} \frac{L}{2}.$$

c) The centripetal acceleration of the center of mass is v_0^2/R, so the equation of motion of the rod reads

$$\mu m g = m v_0 \omega_0,$$

and the minimal coefficient of static friction for the slipless motion is

$$\mu = (v_0 \omega_0)/g = \frac{|v_1^2 - v_2^2|}{2gL}.$$



B) *a)* The figure shows the forces acting on the rod in the plane of the surface when the angle between the rod and the steepest line is φ. Since there is no torque acting on the rod, the angular velocity remains constant: $\omega(t) = \omega_0$. The equation of motion of the center of mass in the direction perpendicular to the rod is

$$m\dot{v} = -mg \sin\alpha \sin\varphi.$$

Taking into account that $\varphi = \varphi_0 + \omega_0 t$ (where φ_0 is the initial angle between the rod and the steepest line), the equation of motion is analogous to the differential equation of a harmonic oscillator. So the magnitude of the velocity of the center of the rod as the function of time is given by

$$v(t) = v(0) + \frac{g \sin\alpha}{\omega_0} \cos(\omega_0 t + \varphi_0),$$

and its direction is always perpendicular to the rod. The x- and y-component of the velocity reads

$$v_x(t) = \left(v(0) + \frac{g \sin\alpha}{\omega_0} \cos(\omega_0 t + \varphi_0)\right) \cos(\omega_0 t + \varphi_0), \tag{1}$$

$$v_y(t) = \left(v(0) + \frac{g \sin\alpha}{\omega_0} \cos(\omega_0 t + \varphi_0)\right) \sin(\omega_0 t + \varphi_0). \tag{2}$$

Taking the time average of the components (so we can eliminate the circling motion of the center of the rod):

$$\langle v_x \rangle = \frac{g \sin\alpha}{\omega_0} \left\langle \cos^2(\omega_0 t + \varphi_0) \right\rangle = \frac{g \sin\alpha}{2\omega_0},$$

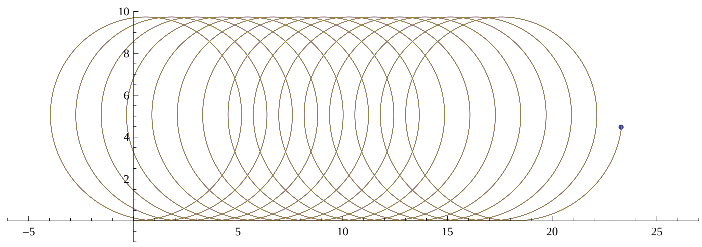
$$\langle v_y \rangle = 0,$$

Now, it is obvious that the drift velocity is

$$v_{\text{drift}} = \frac{g \sin\alpha}{2\omega_0} = \frac{qL \sin\alpha}{2|v_1 - v_2|},$$

and its direction is horizontal (*i.e.* it is perpendicular to the steepest line).

b) We can sketch the orbit of the center of mass:



C) *a)* Equations (1) and (2) are still valid, we only have to fit them to the initial conditions:

$$\mathbf{v}(t) = \left(v_0 + \frac{g \sin\alpha}{\omega_0} (\cos(\omega_0 t) - 1)\right) \begin{pmatrix} \cos(\omega_0 t) \\ \sin(\omega_0 t) \end{pmatrix}$$

b) The velocity of the center of the rod is zero iff

$$\cos(\omega_0 t) = 1 - \frac{v_0 \omega_0}{g \sin\alpha}.$$

Since $-1 \le \cos(\omega_0 t) \le 1$, the condition of the stopping of the center of the rod reads

$$\frac{v_0 \omega_0}{2g \sin\alpha} \le 1.$$

c) We have to distinguish the following two cases: the center of the rod stops for a moment during the motion or not. In the first case, the conservation of mechanical energy tells us that the maximal displacement along the y direction is

$$y_{\max} = \frac{v_0^2}{2g \sin\alpha}.$$

In the second case (when $\frac{v_0\omega_0}{2g\sin\alpha} > 1$), the center of the rod has a finite velocity even at the highest point of its orbit. From the expression of $\mathbf{v}(t)$ one can see, that at the highest point of the orbit the velocity of the center of the rod is

$$v_x(t = \frac{\pi}{\omega_0}) = v_0 - \frac{2g\sin\alpha}{\omega_0},$$

so the conservation of mechanical energy reads

$$\frac{1}{2}mv_0^2 = mgy_{\max}\sin\alpha + \frac{1}{2}mv_x^2,$$

from which one can get the result

$$y_{\max} = \frac{2(v_0\omega_0 - g\sin\alpha)}{\omega_0^2}.$$

D) *a)* In this situation using the equations (1) and (2) we can get the velocity of the rod's center as the function of time:

$$\mathbf{v}(t) = \left(\frac{g\sin\alpha}{\omega_0}\sin(\omega_0 t)\right)\begin{pmatrix} \sin(\omega_0 t) \\ -\cos(\omega_0 t) \end{pmatrix},$$

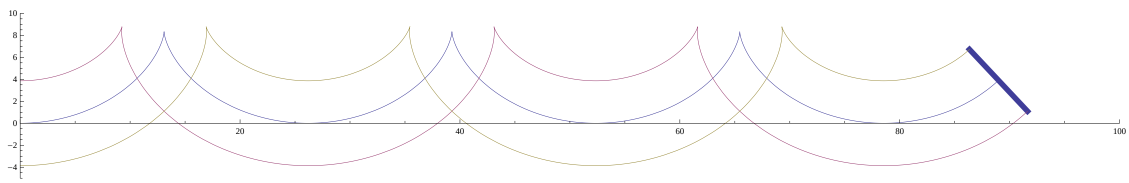
after some trigonometrical manipulations we get:

$$\mathbf{v}(t) = \left(\frac{g\sin\alpha}{2\omega_0}\right)\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \cos(2\omega_0 t) \\ \sin(2\omega_0 t) \end{pmatrix}\right].$$

It can be seen that the endpoint of the velocity vector moves along a circle with radius $v^* = \frac{g\sin\alpha}{2\omega_0}$ and center $(\frac{g\sin\alpha}{2\omega_0}, 0)$. So the center of the rod moves as same as a point at the perimeter of a rolling circle with velocity v^*, which is a cycloid. We can calculate the radius of this 'rolling circle', if we recognize that the period of this motion is π/ω_0:

$$r^* = \frac{1}{2\pi}\frac{\pi}{\omega_0}v^* = \frac{g\sin\alpha}{4\omega_0^2}.$$

The difference between the lowest and highest position of the center of the rod is $\Delta y = 2r^*$ (measured along the steepest line). The orbit of the center and the two endpoints of the rod is shown in the figure.



b) The magnitude of the force of static friction is maximal if the center of the rod is in the lowest possible position during the motion. At this point the centripetal acceleration of the center of mass is $(v^*)^2/r^*$, so the equation of motion is

$$\mu mg\cos\alpha - mg\sin\alpha = m\frac{(v^*)^2}{r^*},$$

so $\mu = 2\mathrm{tg}\alpha$.