# Why Maglev trains levitate

A WPhO problem proposal

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# Problem. Why Maglev trains levitate?

## Introduction

*Maglev* is a technology for magnetic suspension (levitation) and propulsion of trains or other vehicles. Since there is no friction force between the train and the rails Maglev trains reach record velocities approaching 600 km/h.

There are three types of Maglev systems – EMS (Electromagnetic Suspension), EDS (Electrodynamic Suspension), and the experimental Inductrack technology. In this problem you are going to explore the physical principles of Inductrack suspension on a simplified model of a Maglev train. So far the principle of magnetic propulsion will not be considered here since it has a lot of common with the physics of magnetic levitation.

Shown in Figure 1 is a schematic side view of an Inductrack train-car. When at rest or moving at a low speed the car lies with its wheels on the rails like any ordinary train. The car, however, detaches from the rails at a specific takeoff velocity $v_t$ due to the system described below: Two long parallel arrays of permanent cubic-shape magnets of size $a$ = 5 cm each are located at the bottom of the car. The magnetic dipole moments of the neighboring magnets are tilted at an angle of 45° relative to each other (the so called *Halbach array*). As a result a static magnetic wave is produced below each array with components of the magnetic induction given by the equations:

(1) $$B_x = B_0 \exp(-ky)\sin(kx)$$

(2) $$B_y = B_0 \exp(-ky)\cos(kx)$$

where the $x$-coordinate is measured from the rear end of the car in the direction of motion and $y$-coordinate from the bottom of the car in a downward direction. The parameter $k$ is the wavevector of the magnetic wave. The amplitude of the magnetic induction is $B_0$ = 1.4 T.

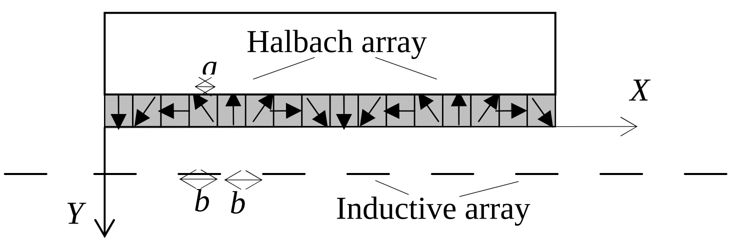


**Figure 1.** *A schematic side view of an Inductrack train-car. The arrows show the directions of magnetic moments of the cubic magnets in the Halbash array. The wheels and the rails of the train are not shown for convenience.*

Two *inductive arrays* of horizontal rectangular wire frames of the same width $a$ as that of the permanent magnets are arranged along the guide way as seen from the top view in Figure 2. The length of the inductive frames as well as the distance between them is $b$.  Each inductive array is located below the corresponding Halbach array at a distance corresponding to the $y$-coordinate of the inductive array (see Figure 1).. It is assumed that: the equations (1) and (2) for the components of the magnetic induction hold true only for those inductive frames, which are situated below the car, while the magnetic induction outside the car area is strictly zero.

*Important note*. In the real Inductrack trains the frames of the inductive array are electrically connected and form a continuous ladder-like array. In order to simplify the theoretical consideration, however, we adopt in this problem the simplified model shown in Figure 2.

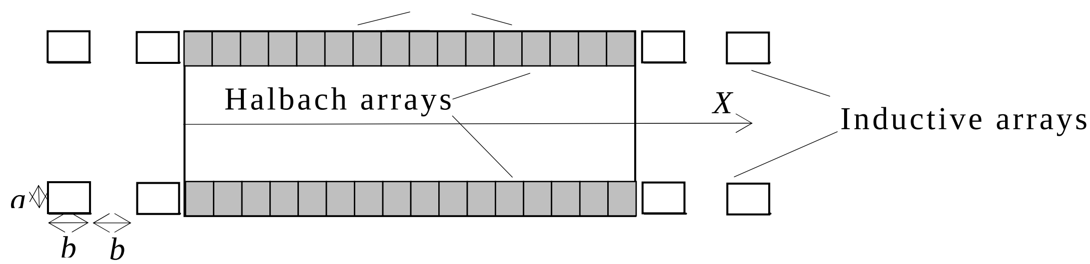


**Figure 2.** *A schematic top-view of an Inductrack train-car.. The wheels and the rails of the train are not shown for convenience.*

***Helpful mathematics***

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos(x) - \cos(y) = 2\sin\left(\frac{y-x}{2}\right)\sin\left(\frac{x+y}{2}\right)$$

## Tasks

In tasks 1–4 assume that the train car is infinitely long, i.e. equations (1) and (2) hold true for all values of the $x$–coordinate.

**Task 1.** Derive a relationship between the wavevector $k$ of the magnetic wave and the lateral size $a$ of the cubic magnets in the Halbach array. Calculate $k$ numerically.

**Task 2.** Suppose that the train moves with a constant velocity $v$ in the positive direction of the $X$-axis. Consider an inductive frame whose center in the initial moment ($t$ = 0) is located below the car area at a point of coordinate $x_c$ relative to the car.

- Derive an expression for the electromotive force (EMF) $E$ induced in the frame as a function of the time $t$ and the parameters already defined.

- What is the circular frequency ω of the induced EMF.

**Task 3.** Each frame in the inductive array is characterized by a self-inductance $L$ and a resistance $R$. The mutual inductance between different frames is negligible compared to $L$. We assume that the current $I$ induced in the frame considered in Task 2 varies with time according to the equation:

(3) $$I(t) = I_0 \sin(\omega t - kx_c + \varphi)$$

- Obtain expressions for the amplitude $I_0$ and the phase-shift φ of the alternative current induced in the frame.

*Important note.* The positive direction of circulation of the induced current in the frame is related to the positive direction of the $Y$-axis according to the right-hand rule.

**Task 4.** Derive formulas for the time-averaged components $\overline{F}_x$ and $\overline{F}_y$ acting on a single inductive frame in terms of velocity $v$, the distance $y$ between the Halbach array and the inductive array and the parameters already defined.. Sketch qualitative graphs of $\overline{F}_x$ and $\overline{F}_y$ versus train velocity $v$ for a fixed value of the distance $y$.

**Task 5.** Consider a train-car of a large but finite length $l$ ($l >> b$).

- Derive expressions for the magnitude of the total lift (vertical) and drag (horizontal) forces $F_L$ and $F_D$ acting on the car.

- What is the minimum aspect ratio $b/a$ for which the lift force attains a maximum magnitude for given values of the distance $y$, velocity $v$, resistance $R$ and the inductance $L$..

**Task 6.** Consider a train-car of a length $l$ = 10 m and a mass $m$ = 10 000 kg. Assume that the aspect ratio $b/a$ corresponds to the optimal value found in Task 5. The inductance and the resistance of the inductive frames are $L = 1.0\times10^{-7}$ H and $R = 1.0\times10^{-5}\ \Omega$ respectively. The acceleration due to gravity is $g = 9.8\ \text{m/s}^2$. Assume that the distance between the Halbach array and the inductive array is $y = 0$ when the train is at rest.

- Obtain an expression and calculate the takeoff velocity $v_t$, i.e. the velocity at which the train detaches from the rails.

- What is the gap $y$ between the train and the rails at an operating velocity of 360 km/h?

# Solution. Why Maglev trains levitate?

**Task 1.** The wavelength $\lambda$ of the magnetic wave corresponds to the spatial period of the Halbach array:

$$\lambda = 8a$$

Therefore, the wavevector of the wave is:

$$\boxed{k = \frac{2\pi}{\lambda} = \frac{\pi}{4a}}$$

or numerically:

$$\boxed{k \approx 15.7\,\text{m}^{-1}}$$

**Task 2.** There are two approaches for obtaining the EMF depending on the choice of the system of reference.

I. A reference system fixed to the Earth

A point, having at time $t$ a coordinate $x$ relative to the Earth, has a "shifted" coordinate $x - vt$ relative to the train. Therefore the components of the magnetic induction relative to the Earth depend on the spatial coordinates and the time according to the equations:

$$B_x(x,t) = B_0 \exp(-ky)\sin(k(x - vt))$$

$$B_y(x,t) = B_0 \exp(-ky)\cos(k(x - vt)) \,.$$

Since relative to the Earth the $x$-coordinate of the given inductive frame spans a fixed interval $x \in [x_c - b/2\,; x_c + b/2]$, the magnetic flux through the frame is:

$$\Phi(t) = a \int_{x_c - b/2}^{x_c + b/2} B_y(x,t)dx = \frac{2B_0 a}{k} \exp(-ky)\sin\left(\frac{kb}{2}\right)\cos(k(x_c - vt))$$

According to the Faraday's law of magnetic induction, the induced EMF is:

$$E = -\frac{d\Phi(t)}{dt} = -2B_0 av \exp(-ky)\sin\left(\frac{kb}{2}\right)\sin(k(x_c - vt))$$

or

$$\boxed{E = 2B_0 av \exp(-ky)\sin\left(\frac{kb}{2}\right)\sin(\omega t - kx_c)}$$

where the circular frequency of the induced EMF is:

$$\boxed{\omega = kv = \frac{\pi v}{4a}}$$

II. A reference system connected to the car

In this system the magnetic wave is static but the inductive frame moves with velocity $-v$ relative to the field. Therefore, the $x$-coordinate of the center of the frame relative to the train is $x = x_c - v\,t$. In that case an EMF is induced along any small element $d\vec{r}$ of the frame:

$$dE = \left[(-\vec{v})\times\vec{B}\right]. d\vec{r}$$

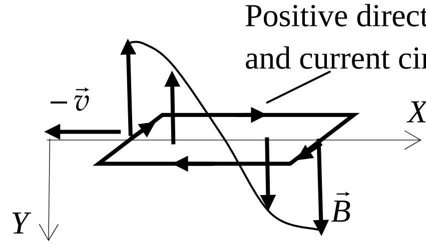


**Figure S1.**

The EMF is induced only along the two sides of the frame, which are perpendicular to the $X$-axis (see Figure S1). Taking into account that the $x$-coordinate of these two sides are $x = x_c - v\,t \pm b/2$, the total EMF in the frame is:

$$E = vB_y(x_c - vt + b/2)a - vB_y(x_c - vt - b/2)a$$

After simplification we obtain the same result as above:

$$\boxed{E = 2B_0 av\exp(-ky)\sin\left(\frac{kb}{2}\right)\sin(\omega t - kx_c)}$$

**Task 3.** The parameters of the induced current could be obtained in two equivalent ways.

I. We set up the Kirchhoff’s equation for the conductive frame:

$$E_0\sin(\omega t - k_c x) - L\frac{dI}{dt} = IR$$

where $E_0 = 2B_0 av\exp(-ky)\sin\left(\frac{kb}{2}\right)$ is the amplitude of the induced EMF. We substitute into that equation the time dependence of the current:

$$E_0 \sin(\omega t - k_c x) - \omega L I_0 \cos(\omega t - k_c x + \varphi) = I_0 \sin(\omega t - k_c x + \varphi) R$$

Consider two moments of time when the phase $\omega t - k_c x$ of the induced EMF is 0 and π/2 respectively. Thus, we obtain a set of two equations:

$$-\omega L I_0 \cos(\varphi) = I_0 R \sin(\varphi)$$

and

$$E_0 + \omega L I_0 \sin(\varphi) = I_0 R \cos(\varphi)\,.$$

Finally we get:

$$\boxed{\tan(\varphi) = -\frac{\omega L}{R} = -\frac{k v L}{R}}$$

and

$$\boxed{I_0 = \frac{E_0}{\sqrt{R^2 + (\omega L)^2}} = \frac{2 B_0 a v \exp(-ky)}{\sqrt{R^2 + (k v L)^2}} \sin\left(\frac{kb}{2}\right)}.$$

II. The inductive frame is equivalent to a circuit of series-connected inductance $L$ and an Ohmic resistance $R$ as shown in Figure S2. We consider the rotating-vector diagram for the current $I$ and voltage drops $U_L$ and $U_R$ across the inductance and the resistance respectively (see Figure S2). The amplitudes of the voltage drops are connected to the amplitude $I_0$ of the current through:

$$U_L = \omega L I_0$$

and

$$U_R = R I_0\,.$$

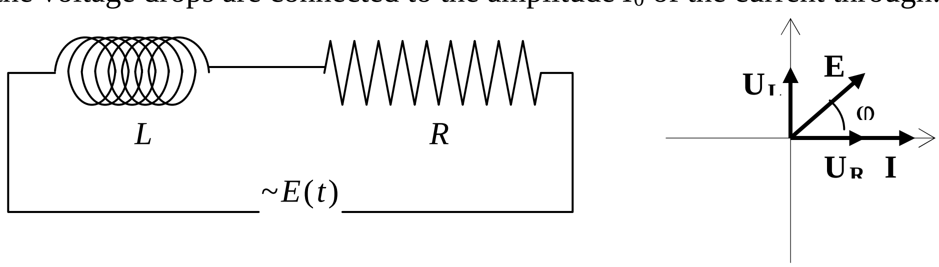


**Figure S2.** *An equivalent circuit and a vector diagram for the alternative current and EMF in the inductive frame.*

It follows from the diagram that the amplitude $E_0$ of the EMF is given by:

$$E_0 = \sqrt{U_R^2 + U_L^2} = \sqrt{R^2 + (\omega L)^2}\, I_0$$

Therefore, the amplitude of the current is:

$$\boxed{I_0 = \frac{E_0}{\sqrt{R^2+(\omega L)^2}} = \frac{2B_0 a v \exp(-ky)}{\sqrt{R^2+(kvL)^2}}\sin\left(\frac{kb}{2}\right)}.$$

It is evident from Figure S2 that:

$$\boxed{\tan(\varphi) = -\frac{\omega L}{R} = -\frac{kvL}{R}}$$

The "–" sign means that the current is in retard relative to the induced EMF.

**Task 4.** As it follows from Figure S1 the vertical component of the force acting on the inductive frame is:

$$F_y(t) = I(t)a\{B_x(x_c - b/2) - B_x(x_c + b/2)\} = -2I_0\sin(\omega t - kx_c + \varphi)aB_0\sin\left(\frac{kb}{2}\right)\cos(\omega t - kx_c) =$$

$$= 2I_0 B_0 a\exp(-ky)\sin\left(\frac{kb}{2}\right)\sin(\omega t - kx_c + \varphi)\sin(\omega t - kx_c - \pi/2)$$

The time-average of that function is:

$$\overline{F}_y = I_0 B_0 a\exp(-ky)\sin\left(\frac{kb}{2}\right)\cos(\varphi + \pi/2) = -I_0 B_0 a\exp(-ky)\sin\left(\frac{kb}{2}\right)\sin(\varphi)$$

By taking into account the expressions for $I_0$, and $\varphi$, and the relation $\sin\varphi = \frac{\tan\varphi}{\sqrt{1+\tan^2\varphi}}$ we finally obtain:

$$\boxed{\overline{F}_y = I_0 = \frac{E_0}{\sqrt{R^2+(\omega L)^2}} = \frac{2B_0^2 a^2 v^2 kL\exp(-2ky)}{R^2+(kvL)^2}\sin^2\left(\frac{kb}{2}\right)}$$

Concerning the horizontal component of the force, we may take an advantage by using the relation

$$\overline{F}_x = \frac{\overline{P}}{v}$$

where

$$\overline{P} = \frac{1}{2}I_0^2 R$$

is the average power dissipated by the current in the frame. By using the expression for $I_0$ we obtain:

$$\boxed{\overline{F}_x = \frac{2B_0^2 a^2 vR\exp(-2ky)}{R^2 + (kvL)^2}\sin^2\left(\frac{kb}{2}\right)}$$

Qualitative graphs of the two functions are shown in Figure S3.

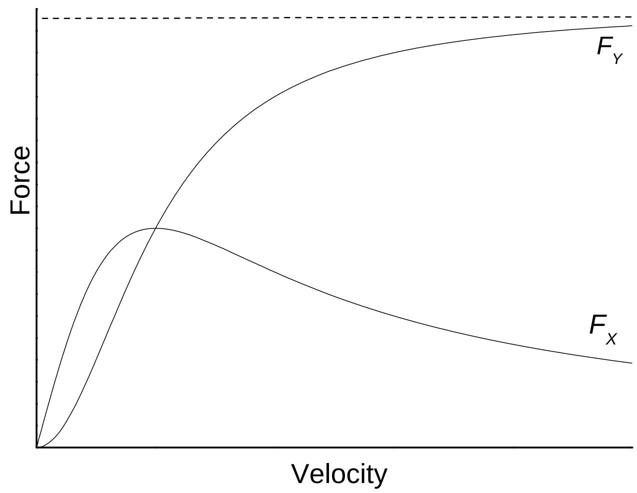


**Figure S3.**

Points should be awarded for the following details:

- A clear indication that the vertical component of the force tends to a specific limit as the velocity increases.
- The horizontal component of the force has a maximum for a specific velocity.

**Task 5.** According to the Newton's third law any single inductive frame exercises on the train a vertical force $(-\overline{F}_y) < 0$, i.e. a force directed upward, against the gravity. Similarly, any frame acts on the car with a horizontal force $(-\overline{F}_x) < 0$ in the direction opposite to the direction of motion. Since there are:

$$N = 2\times l/(2b) = l/b$$

frames below the car area, the magnitudes of the lift and the drag forces are respectively:

$$\boxed{F_L = \frac{2B_0^2 a^2 v^2 kLl \exp(-2ky)}{(R^2+(kvL)^2)b}\sin^2\left(\frac{kb}{2}\right)} \quad \text{and} \quad \boxed{F_D = \frac{2B_0^2 a^2 vRl \exp(-2ky)}{(R^2+(kvL)^2)b}\sin^2\left(\frac{kb}{2}\right)}$$

By introducing the aspect ratio $x = b/a$ and noticing that $k = \pi/(4a)$ the problem of maximization of the lift force is equivalent to finding a maximum of the function:

$$f(x) = \frac{\sin^2(\pi x/8)}{x}$$

After taking a first derivative of $f(x)$ the condition for a maximum reduces to solving the equation:

$$\tan(\pi x/8) = \pi x/4$$

Since in the point of intercept of the two sides of the equation, the tangent function has bigger slope than the linear function, it is convenient to rewrite the equation in a form that allows an iterative solution:

$$x_{n+1} = 2.5465\arctan(0.7854 x_n)$$

By starting with a trial value, e.g. $x_0 = 1$, we obtain the following convergent series of approximations:

| $n$ | $x_n$ | $n$ | $x_n$ |
|---|---|---|---|
| 1 | 1.6954 | 7 | 2.9658 |
| 2 | 2.3597 | 8 | 2.9674 |
| 3 | 2.7400 | 9 | 2.9679 |
| 4 | 2.8923 | 10 | 2.9680 |
| 5 | 2.9440 | 11 | 2.9681 |
| 6 | 2.9606 | 12 | 2.9681 |

Therefore, up to a precision of 4 significant digits, the optimal aspect ratio is:

$$\boxed{b/a = 2.968}$$

**Task 6.** For the optimal aspect ratio and by taking into account that $ak = \pi/4$ the expression for the lift force simplifies:

$$F_L = 0.447\frac{B_0^2 v^2 Ll \exp(-2ky)}{R^2+(kvL)^2}$$

For $y = 0$ the train detaches at a velocity such that $F_L = mg$, i.e.

$$\boxed{v_t = R\sqrt{\frac{mg}{0.447B_0^2Ll - mg(kL)^2}}}$$

For the parameters specified:

$$\boxed{v_t = 3.93\,\text{m/s} \approx 14\,\text{km/h}}$$

For greater velocities the train levitates at a distance *y* above the rails such that the equilibrium condition is satisfied:

$$0.447\frac{B_0^2v^2Ll\exp(-2ky)}{R^2+(kvL)^2} = mg$$

Therefore, the gap between the wheels and the rails at a velocity of 360 km/h (100 m/s) is:

$$\boxed{y = \frac{1}{2k}\ln\left(\frac{0.447B_0^2v^2Ll}{mg\left(R^2+(kvL)^2\right)}\right) = 21\,\text{cm}}$$