Mirage

Prof. Ruo Peng WANG

School of Physics

Peking University

Beijing 100871

P.R.China

rpwang@pku.edu.cn

rpwang@cis.pku.edu.cn

Mirage

The refractive index of the air varies with temperature. Cold air is denser than warm air and has therefore a greater refractive index. Thus a temperature gradient in the atmosphere is always associated with a gradient of the refractive index. Under certain conditions, this gradient of the refractive index could be strong enough so that light rays are bent to produce a displaced image of distant objects. This amazing phenomenon is called as “mirage”.
Mirages can be categorized as "inferior" and "superior". Inferior mirages can be seen on deserts and highways, and superior mirages occur over the sea.

Fig. 1a A superior mirage

Fig. 1b An inferior mirage

To describe in detail the phenomenon of the mirage, we need to analyze the path of light rays in media with a refractive index gradient.

1 The path of a light ray and the trajectory of a mass point

Media with a refractive index gradient can be treated as being formed from thin homogeneous layers with different refractive indexes. The path of a light ray can be determined then by analyzing refractions of the light ray at interfaces between these thin homogeneous layers. But there exists a more convenient way for determining the path of a light ray in media with a refractive index gradient. Instead of analyzing the propagation of the light, one may study the motion of a mass point that moves along the path of the light ray, driven by a conservative force. The potential energy of the conservative force field depends on the distribution the refractive index. Once the potential energy is established, one may study the motion of the mass point by using well developed tools in classical mechanics, and find the trajectory of a mass point which is also the path of the light ray.

1.1 Refraction of a light ray at the interface between a medium with a refractive index n_1 and a medium with a refractive index n_2 is shown in Fig. 2a. The angles i_1 and i_2 obey the Snell’s law

$$n_1 \sin i_1 = n_2 \sin i_2$$

The path of a mass point in a conservative force field is illustrated in Fig.2b. The speed of the mass point is v_1 in the region with the potential energy E_{p1}, and v_2 in the region with the potential energy E_{p2}. Find expressions for v_1 and v_2 so that the relation $n_1 \sin i_1 = n_2 \sin i_2$ held also for the trajectory of the mass point. You may use an arbitrary constant v_0 with the dimension of speed in your expressions. (0.2 point)

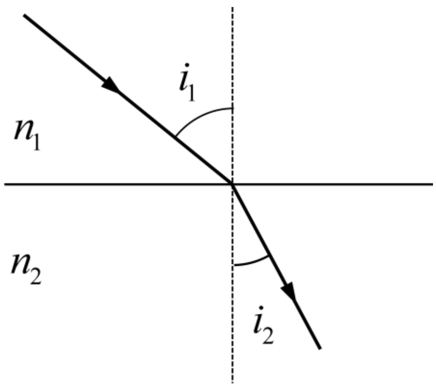


Fig. 2a Refraction of a light ray

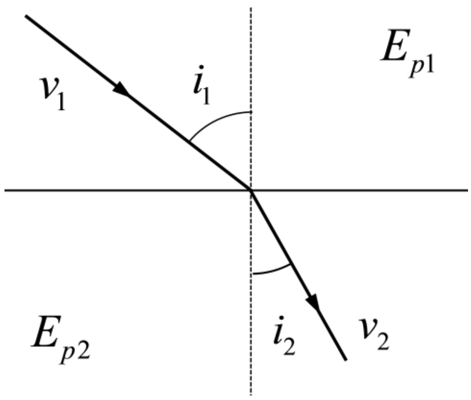


Fig. 2b Deviation of a mass point in a conservative force field

1.2 Assume that the mass of the mass point is m, and the total energy of the mass point equals to 0. Find the potential energies E_{p1} and E_{p2}. (0.2 point)

1.3 The trajectory of a mass point with a mass m in a conservative force field is same as the path of a light ray in a medium with a refractive index $n(\vec{r})$ which is a function of the position. The total energy of this mass point is zero. Find expressions for the potential energy $E_p(\vec{r})$ of the conservative force field and the speed of the mass point $v(\vec{r})$. (0.4 point)

1.4 To describe the motion of a mass point, one expresses the position of the mass point as a function of time: $\vec{r}(t)$. To describe the trajectory of a mass point, one needs to express the position of the mass point as a function of the distance s traveled by the mass point from the start point of the trajectory to the current position: $\vec{r}(s)$.

Derive a differential equation for the trajectory $\vec{r}(s)$ of a mass point with a mass m and a total energy E that moves in a conservative force field. (0.6 point)

Hint: in a conservative force field, $\vec{r}(t)$ satisfies Newton's second law:

$$m\frac{d^2\vec{r}}{dt^2}=-\nabla E_p \text{, and } \frac{ds}{dt}=v(\vec{r})$$

1.5 Derive the light ray equation (a differential equation for the path of a light ray) by using the results obtained in 1.3 and 1.4. (0.6 point)

Hint: $\nabla f^2(\vec{r})=2f(\vec{r})\nabla f(\vec{r})$

2 The inferior mirage

When an inferior mirage appears, an image of a distant object can be seen under the real object. A direct image of that object is seen also because some of the light rays that enter the eye in a straight line without being refracted. The double image seems to be that of the object and its upside-down reflection in water. For exhausted travelers in the desert it seems like that there is a lake of water in their front.

An inferior mirage occurs when a strong positive gradient of refractive index is present near the ground. We use the following model to describe the variation of the refractive index of the air with elevation:

$$n^2(z)=\begin{cases} n_0^2+\alpha z & \text{for } h\ge z\ge 0 \\ n_0^2+\alpha h & \text{for } z>h \end{cases}$$

with $n_0^2-1\ll 1$ and $\alpha h\ll 1$.

2.1 Find the path of a light ray (i.e. z as a function of x) that enters in an observer's eye at an angle θ (see Fig. 3). The height of the observer's eye is H. (1.9 point)

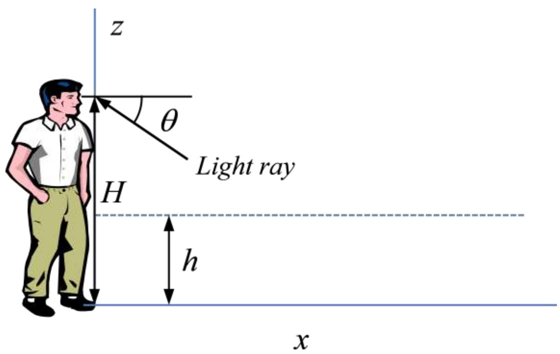


Fig.3 The geometry for analyzing an inferior mirage

2.2 Due to the inferior mirage, the observer can see an inverted image of the upper part of a camel at a large distance. The parameters for the refractive index of the air are $\alpha = 3.0\times 10^{-5} m^{-1}$, $h = 0.50m$. The height of the observer's eye is $H = 1.5m$, and the height of the camel is $l = 2.2m$. Find the minim value for the distance D_m between the observer and the camel. You use the approximation $n_0^2 \approx 1$.

(0.7 point)

2.3 As a result of the light refraction in the region with refractive index gradient, the observer cannot see the lower part of the camel's legs. Find the height of the lowest point on the camel that can be seen by the observer l_m. (0.7 point)

2.4 Find the distance between the observer and the imaginary lake of water d.

(0.3 point)

3 The superior mirage

When a superior mirage occurs, light rays that were originally directed above the line of sight will reach the observer's eye. Thus, an object ordinarily below the horizon will be apparently above the horizon.

A superior mirage occurs when a negative gradient of refractive index is present over a body of water or over large sheets of ice. We use the following model to describe the refractive index of the atmosphere (see Fig. 4):

$$n^2(r) = \begin{cases} n_0^2[1-\beta(r-r_0)] & for \;\; b \ge (r-r_0) \ge 0 \\ n_0^2(1-\beta b) & for \;\; (r-r_0) > b \end{cases}$$

where r_0 is the radius of the earth, $b \ll r_0$ and $\beta b \ll 1$..

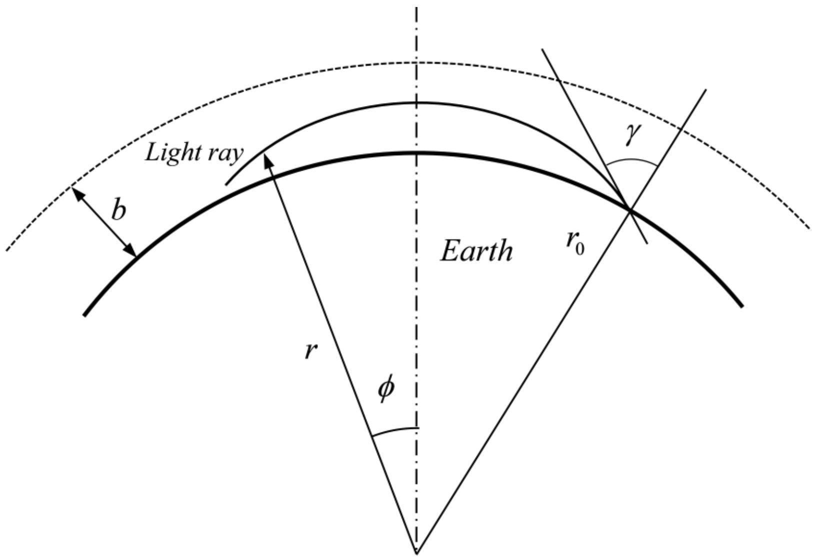


Fig.4 The geometry for analyzing a superior mirage

3.1 Find the path of a light ray (*i.e.* r as a function of ϕ) within the range $b \geq (r - r_0) \geq 0$. Use the angle between the light ray and the vertical direction γ at the sea level as a parameter of the path. (1.4 point)

Hint: The trajectory of a mass point with a mass point with a mass m, an angular momentum L and a total energy E in a conservative force field with

$$E_p = -\frac{mA}{r} + E_0$$

is

$$r = \frac{\rho}{1 - \varepsilon \cos\phi}$$

where

$$\rho = \frac{L^2}{m^2 A}, \quad \varepsilon = \sqrt{1 + \frac{2(E - E_0)L^2}{m^3 A^2}}$$

We have also

$$\frac{1}{r} \approx \frac{1}{r_0} - \frac{r - r_0}{r_0^2} \quad if \; |r - r_0| \ll r_0$$

3.2 Find the minim value β_m of β at which the superior mirage occurs. Use the following values for n_0 and r_0: $n_0 \approx 1$, $r_0 = 6.4 \times 10^6 m$. (0.8 point)

3.3 Under certain atmospheric condition, $b = 100m$ and $\beta = 6.0 \times 10^{-7} m^{-1}$. Calculate the largest distance D_M at which the surface of the sea can be seen by an observer at an altitude $y = 10m$. (*n.b.*: y is the altitude of the observer's eye)

(1.4 point)

3.4 For a comparison, calculate the largest distance D'_M at which the surface of the sea can be seen by an observer at the same altitude $y = 10m$, when the refractive index of the air is constant . (0.3 point)

3.5 Calculate the angular difference $\Delta\vartheta$ between the apparent horizon when a superior mirage occurs as described in 3.3 and the apparent horizon in a normal day as described in 3.4 , seen at the same altitude $y = 10m$. (0.5 point)

Useful formula:

$$\cos\phi \approx 1 - \frac{1}{2}\phi^2 \quad \text{for} \quad \phi \ll 1.$$

ANSWER SHEET

1 The path of a light ray and the trajectory of a mass point

1.1	$v_1 =$ $v_2 =$	0.2
1.2	$E_{p1} =$ $E_{p2} =$	0.2
1.3	$v(\vec{r}) =$ $E_p(\vec{r}) =$	0.4
1.4		0.6
1.5		0.6

2 The inferior mirage

2.1		1.9
2.2	$D_m =$	0.7
2.3	$l_m =$	0.7

2.4	$d =$	0.3

3 The superior mirage

3.1		1.4
3.2	$\beta_m =$	0.8
3.3	$D_M =$	1.4
3.4	$D'_M =$	0.3
3.5	$\Delta\vartheta =$	0.5

Solution

1 The path of a light ray and the trajectory of a mass point

1.1 The component of the conservative force parallel to the interface is null, thus

$$v_1 \sin i_1 = v_2 \sin i_2$$

This relation becomes

$$n_1 \sin i_1 = n_2 \sin i_2$$

if

$$v_1 = n_1 v_0 \quad and \quad v_2 = n_2 v_0 \qquad \text{(0.2 point)}$$

1.2 $E_{p1} = 0 - \frac{1}{2} m v_1^2 = -\frac{1}{2} m n_1^2 v_0^2$ and $E_{p2} = 0 - \frac{1}{2} m v_2^2 = -\frac{1}{2} m n_2^2 v_0^2$ (0.2 point)

1.3 $v(\vec{r}) = n(\vec{r}) v_0$ (0.2 point)

and

$$E_p(\vec{r}) = 0 - \frac{1}{2} m v^2(\vec{r}) = -\frac{1}{2} m n^2(\vec{r}) v_0^2 \qquad \text{(0.2 point)}$$

1.4 $\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = v\frac{d}{ds}$, thus $\frac{d^2\vec{r}}{dt^2} = v\frac{d}{ds}\left(v\frac{d\vec{r}}{ds}\right)$. (0.2 point)

The equation for the trajectory is

$$mv\frac{d}{ds}\left(v\frac{d\vec{r}}{ds}\right) = -\nabla E_p \quad \text{with} \quad v = \sqrt{\frac{E - E_p}{2m}} \qquad \text{(0.4 point)}$$

1.5 $mv\frac{d}{ds}\left(v\frac{d\vec{r}}{ds}\right) = -\nabla E_p$ becomes $mnv_0\frac{d}{ds}\left(nv_0\frac{d\vec{r}}{ds}\right) = -\nabla\left(-\frac{1}{2}mn^2v_0^2\right)$

(0.2 point)

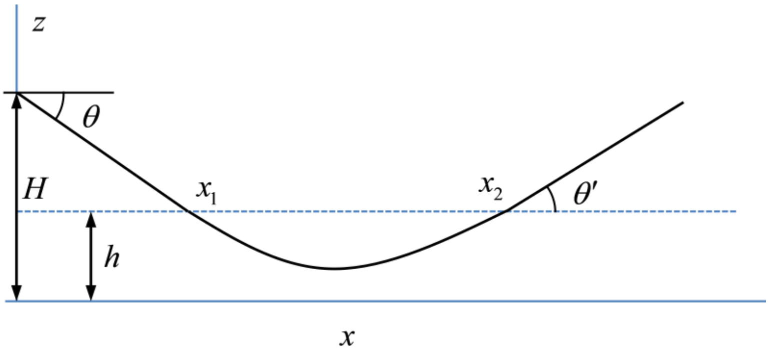
But $-\nabla\left(-\frac{1}{2}mn^2v_0^2\right) = mv_0^2 n\nabla n$ and $mnv_0\frac{d}{ds}\left(nv_0\frac{d\vec{r}}{ds}\right) = mnv_0^2\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right)$

Therefore

$$\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \nabla n \qquad \text{(0.4 point)}$$

2 The inferior mirage

2.1 The path of light ray is a straight line for $z > h$.



Thus

$$z = H - x\tan\theta \quad for \quad x < x_1 \qquad \text{(0.2 point)}$$

with

$$x_1 = (H-h)\cot\theta \qquad \text{(0.1 point)}$$

For $x_2 \geq x \geq x_1$, the corresponding potential energy is

$$E_p(z) = -\frac{1}{2}mv_0^2n_0^2 - \frac{1}{2}mv_0^2\alpha z \qquad \text{(0.2 point)}$$

The equation of motion of the corresponding mass point is then

$$\begin{cases} m\dfrac{d^2z}{dt^2} = \dfrac{1}{2}mv_0^2\alpha \\ m\dfrac{d^2x}{dt^2} = 0 \end{cases} \qquad \text{(0.2 point)}$$

With the initial condition

$$\left.\frac{dx}{dt}\right|_{t=0} = n(h)v_0\cos\theta \quad and \quad \left.\frac{dz}{dt}\right|_{t=0} = -n(h)v_0\sin\theta \qquad \text{(0.2 point)}$$

So we have

$$\begin{cases} z = \dfrac{1}{4}v_0^2\alpha t^2 - n(h)v_0t\sin\theta + h \\ x = n(h)v_0t\cos\theta + x_1 \end{cases} \qquad \text{(0.2 point)}$$

By eliminating t in above equations, we obtain

$$z = \frac{\alpha}{4n^2(h)\cos^2\theta}(x-x_1)^2 - x\tan\theta + H \qquad \text{(0.2 point)}$$

This expression can also be written as

$$z = \frac{\alpha}{4n^2(h)\cos^2\theta}\left(x-(H-h)\cot\theta-\frac{2n^2(h)\cos\theta\sin\theta}{\alpha}\right)^2-\frac{n^2(h)\sin^2\theta}{\alpha}+h$$

This path may reach in the region with $x > x_2$ if

$$-\frac{n^2(h)\sin^2\theta}{\alpha}+h \ge 0$$

or

$$\sin\theta \le \frac{\sqrt{\alpha h}}{n(h)} \qquad \text{(0.2 point)}$$

The value of x_2 can determined by the following condition

$$h = \frac{\alpha}{4n^2(h)\cos^2\theta}\left(x_2-(H-h)\cot\theta-\frac{2n^2(h)\cos\theta\sin\theta}{\alpha}\right)^2-\frac{n^2(h)\sin^2\theta}{\alpha}+h$$

We find

$$x_2 = (H-h)\cot\theta+\frac{4n^2(h)\cos\theta\sin\theta}{\alpha} \qquad \text{(0.1 point)}$$

We also have

$$\tan\theta' = \left.\frac{dz}{dx}\right|_{x=x_2} = \tan\theta \qquad \text{(0.1 point)}$$

Then we have for $x > x_2$

$$z = (x-x_2)\tan\theta+h \qquad \text{(0.2 point)}$$

We can express the path of light ray as

$$z = \begin{cases} H - x\tan\theta & for \quad x < x_1 \\ \dfrac{\alpha}{4n^2(h)\cos^2\theta}\left(x-\dfrac{1}{2}(x_1+x_2)\right)^2-\dfrac{n^2(h)\sin^2\theta}{\alpha}+h & for \quad x_2 \ge x \ge x_1 \\ (x-x_2)\tan\theta+h & for \quad x > x_2 \end{cases}$$

with

$$x_1 = (H-h)\cot\theta$$

and

$$x_2 = (H-h)\cot\theta+\frac{4n^2(h)\cos\theta\sin\theta}{\alpha}$$

2.2 Let x_c be the coordinate of the camel. To see the inverted image of the upper part of a camel we must have

$$l = \left(x_c - x_2\right)\tan\theta + h \qquad \text{(0.1 point)}$$

and

$$\sin\theta \le \frac{\sqrt{\alpha h}}{n(h)} \qquad \text{(0.1 point)}$$

Substituting the expression for x_2 we find

$$x_c = (H + l - 2h)\cot\theta + \frac{4n^2(h)\cos\theta\sin\theta}{\alpha} \qquad \text{(0.2 point)}$$

For small θ, x_c decreases with θ, so x_c is minim for

$$\theta = \theta_M = \arcsin\left(\frac{\sqrt{\alpha h}}{n(h)}\right) = 3.87\times 10^{-3}\,rad$$

Thus

$$\begin{aligned} D_m &= (H + l - 2h)\cot\theta_M + \frac{4n^2(h)\cos\theta_M \sin\theta_M}{\alpha} \\ &\approx \frac{H + l - 2h}{\sqrt{\alpha h}} + 4\frac{\sqrt{\alpha h}}{\alpha} \qquad \text{(0.3 point)} \\ &= \frac{H + l + 2h}{\sqrt{\alpha h}} \\ &= 1.21\times 10^3\, m \end{aligned}$$

2.3 Let z_c be the height of a point on the camel. If the observer can see this point, then we must have

$$z_c = \frac{\alpha}{4n^2(h)\cos^2\theta}(D_m - x_1)^2 - D_m\tan\theta + H \qquad \text{(0.1 point)}$$

with $\theta < \theta_M$. As $\theta < \theta_M \ll 1$, so $\cos\theta \approx 1$. We also have $n(h) \approx 1$ and

$$\tan\theta = \frac{H - h}{x_1} \qquad \text{(0.1 point)}$$

Let $y = \dfrac{D_m - x_1}{D_m}$, we have then

$$z_c = \frac{\alpha}{4} D_m^2 y^2 - \frac{H-h}{1-y} + H$$

To find to minim of z_c we need to calculate the derivative $\frac{dz_c}{dy}$

$$\frac{dz_c}{dy} = \frac{\alpha}{2} D_m^2 y - \frac{H-h}{(1-y)^2} \qquad \text{(0.1 point)}$$

The condition for minim z_c is then

$$y = \frac{2(H-h)}{D_m^2 \alpha (1-y)^2} = \frac{0.0455}{(1-y)^2}$$

This equation can be solved numerically. Let $y_1 = 0.0455$ and

$$y_{n+1} = \frac{0.0455}{(1-y_n)^2}$$

We found

n	1	2	3	4	5
y_n	0. 0455	0. 0499	0. 0504	0. 0505	0. 0505

Thus z_c reaches minim at $y_m = 0.0505$

$$l_m = \frac{\alpha}{4} D_m^2 y_m^2 - \frac{H-h}{1-y_m} + H = 0.475m \qquad \text{(0.4 point)}$$

2.4 If a refracted light ray enters the observer's eye at the angle θ, the observer will consider the ray being reflected from a water surface at a distance equal to $H\cot\theta$. The minim value of this distance is the distance between the observer and the "lake". Thus

$$d = H\cot\theta_M \approx \frac{H}{\sqrt{\alpha h}} = 386m \qquad \text{(0.3 point)}$$

3 The superior mirage

3.1 For $b \ge (r - r_0) \ge 0$

$$\begin{aligned} E_p(r) &= -\frac{1}{2} m v_0^2 n_0^2 + \frac{1}{2} m v_0^2 n_0^2 \beta (r - r_0) \\ &\approx -\frac{mA}{r} + E_0 \end{aligned} \qquad \text{(0.1 point)}$$

with

$$A = \frac{1}{2} v_0^2 n_0^2 r_0^2 \beta \qquad \text{(0.2 point)}$$

and

$$E_0 = \frac{mA}{r_0} - \frac{1}{2} m v_0^2 n_0^2 \qquad \text{(0.2 point)}$$

The angular momentum equals to

$$L = m v_0 n_0 r_0 \sin\gamma \qquad \text{(0.2 point)}$$

And the total energy $E = 0$. Thus the path is given by

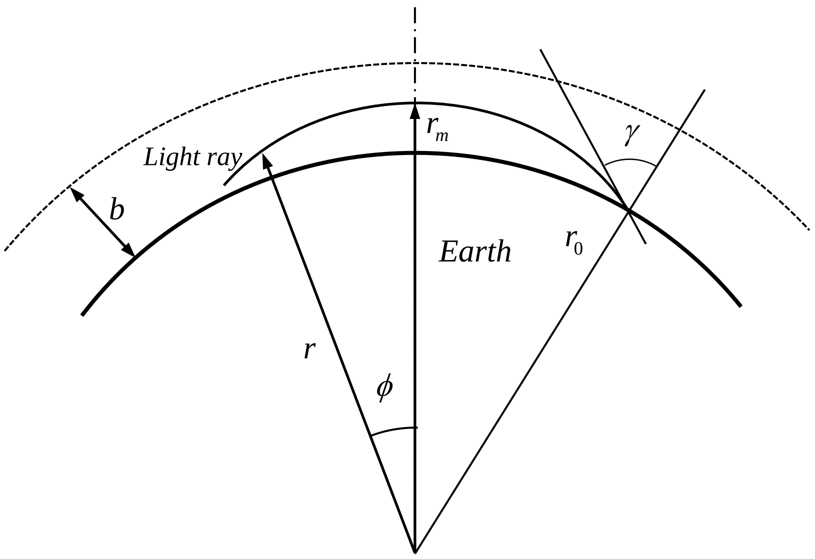
$$r = \frac{\rho}{1 - \varepsilon \cos\phi} \qquad \text{(0.1 point)}$$

with

$$\rho = \frac{2}{\beta} \sin^2 \gamma \qquad \text{(0.3 point)}$$

and

$$\varepsilon = \sqrt{1 - 4\beta^{-2} r_0^{-2} (r_0 \beta - 1) \sin^2 \gamma} \qquad \text{(0.3 point)}$$



3.2 Let r_m be the maxim value of r. When the superior mirage occurs, we have $r_m \le r_0 + b$. The angular momentum is constant

$$L = m v_0 n_0 r_0 \sin\gamma = m v_0 n(r_m) r_m \qquad \text{(0.2 point)}$$

Thus

$$n_0 r_0 \geq n_0 r_0 \sin\gamma = n(r_m) r_m$$

That implies

$$\frac{r_0^2}{r_m^2} \geq 1 - \beta(r_m - r_0)$$

But

$$\frac{r_0^2}{r_m^2} \approx 1 - \frac{2(r_m - r_0)}{r_0}$$

So we must have

$$\beta \geq \frac{2}{r_0} \qquad \text{(0.4 point)}$$

Therefore

$$\beta_m = \frac{2}{r_0} = 3.1\times 10^{-7} m^{-1} \qquad \text{(0.2 point)}$$

3.3 The maxim distance D_M is given by the maxim values of ϕ_1 and ϕ_2

$$D_M = r_0(\phi_{1M} + \phi_{2M}) \qquad \text{(0.1 point)}$$

According to expression for the path of light ray, we have

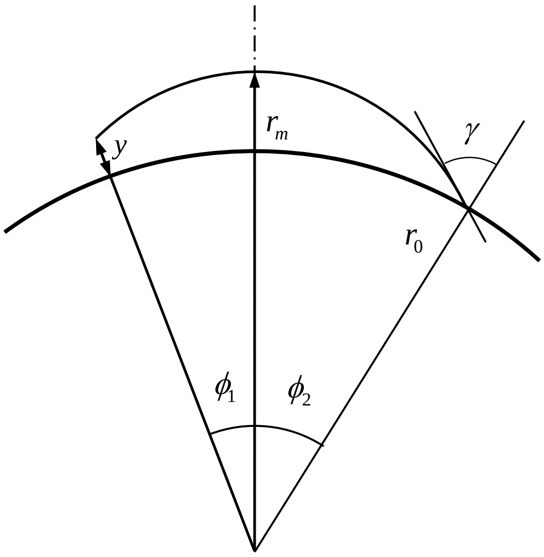
$$r_0 = \frac{\rho}{1-\varepsilon\cos\phi_2} \qquad \text{(0.1 point)}$$

$$r_0 + y = \frac{\rho}{1-\varepsilon\cos\phi_1}$$

(0.1 point)

and

$$r_m = \frac{\rho}{1-\varepsilon} \qquad \text{(0.1 point)}$$



From above relations, we obtain

$$\cos\phi_1 = \frac{1}{\varepsilon}\left(1 - \frac{\rho}{r_0 + y}\right), \quad \cos\phi_2 = \frac{1}{\varepsilon}\left(1 - \frac{\rho}{r_0}\right) \quad and \quad 1 = \frac{1}{\varepsilon}\left(1 - \frac{\rho}{r_m}\right)$$

By eliminating ε we find

$$\cos\phi_2 = \left(1-\frac{\rho}{r_0}\right)\left(1-\frac{\rho}{r_m}\right)^{-1}$$

$$\approx 1-\left(\frac{\rho}{r_0}-\frac{\rho}{r_m}\right)\left(1-\frac{\rho}{r_0}\right)^{-1}$$

$$\approx 1-(r_m-r_0)\frac{\rho}{r_0^2}\left(1-\frac{\rho}{r_0}\right)^{-1}$$

Thus

$$\phi_2 \approx \sqrt{\frac{2\rho(r_m-r_0)}{r_0(r_0-\rho)}} = \sqrt{\frac{4(r_m-r_0)\sin^2\gamma}{r_0(\beta r_0-2\sin^2\gamma)}} \approx 2\sqrt{\frac{r_m-r_0}{r_0(\beta r_0-2)}}$$

and

$$\phi_{2M} \approx 2\sqrt{\frac{b}{r_0(\beta r_0-2)}} \qquad \text{(0.4 point)}$$

Similarly

$$\phi_1 \approx 2\sqrt{\frac{r_m-r_0-y}{r_0(\beta r_0-2)}}$$

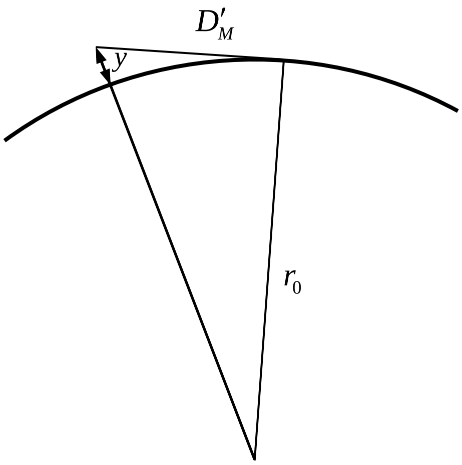
and

$$\phi_{1M} \approx 2\sqrt{\frac{b-y}{r_0(\beta r_0-2)}} \qquad \text{(0.4 point)}$$

Therefore

$$\begin{aligned} D_M &= r_0(\phi_{1M}+\phi_{2M}) \\ &= 2\sqrt{\frac{r_0(b-y)}{\beta r_0-2}}+2\sqrt{\frac{r_0 b}{\beta r_0-2}} \qquad \text{(0.2 point)} \\ &= 7.3\times10^4 m \end{aligned}$$

3.4



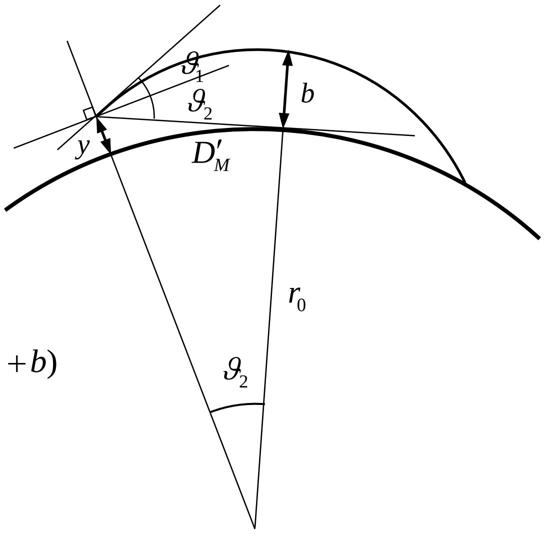
$$\left(D'_M\right)^2 + r_0^2 = \left(r_0+y\right)^2$$

Thus

$$D'_M = \sqrt{\left(r_0+y\right)^2 - r_0^2} \approx \sqrt{2r_0 y} = 1.1\times10^4 m$$

(0.3 point)

3.5 we have



$$\Delta\vartheta = \vartheta_1 + \vartheta_2$$

where

$$\vartheta_2 = \frac{D'_M}{r_0} = \sqrt{\frac{2y}{r_0}} \qquad \text{(0.1 point)}$$

and

$$(r_0 + y)n(r_0 + y)\cos\vartheta_1 = (r_0 + b)n(r_0 + b)$$

Thus

$$\cos\vartheta_1 = 1 - \frac{1}{2}(\beta r_0 - 2)\frac{b-y}{r_0}$$

So

$$\vartheta_1 = \sqrt{(\beta r_0 - 2)\frac{b-y}{r_0}} \qquad \text{(0.3 point)}$$

Therefore

$$\begin{aligned}\Delta\vartheta &= \sqrt{(\beta r_0 - 2)\frac{b-y}{r_0}} + \sqrt{\frac{2y}{r_0}} \\ &= 6.9\times10^{-3}\,rad \qquad \text{(0.1 point)} \\ &= 0.40^\circ\end{aligned}$$