Problem 5 The Rainbow

Deadline May, 31 2012

The rainbow is a spectacular show of the dispersion phenomenon. There are many beautiful tales about the rainbow, but here we will look at the rainbow from a point of view of Physics. Just after rain, there are still a lot of tiny water droplets in the air. If the sun appears from behind the clouds at this time, then the rainbow can be formed from the sunlight refracted and reflected by these tiny water droplets. Water droplets in the air may have different shapes. Only spherical water droplets make contribution to the formation of the rainbow. When falling down through the air, the shape of a water droplet depends on its size. During that process, the surface tension of the water tends to minimize the surface of the droplet and make the droplet spherical, but at the same time, the weight of the droplet and the air resistant force make the water droplet to deform from the spherical shape. The surface tension predominates in smaller water droplets, so smaller water droplets are spherical. For larger water droplets, the weight of the droplet and the air resistant force are stronger than the surface tension, therefore larger water droplets are non-spherical.

Figure 1: The Rainbow

Part 1. The maximal diameter of a spherical water droplet

Consider now a spherical water droplet that falls down through the air with a constant speed. The air resistant force at each point on the surface of the water droplet is tangent to the surface. To simplify the analysis, we assume that the air resistant force is homogeneously distributed over the whole surface of the spherical water droplet (Fig. 2).

1.A. Calculate the air resistant force per unit water droplet surface f . Express your result in terms of the density of the water ρ, the gravitational acceleration g and the diameter of the water droplet D.

1.B. Calculate the horizontal component F_a of the air resistant force acting on a quarter of the sphere in the lower part of the water droplet as shown in Fig. 3. Express your result in terms of ρ, g and D.

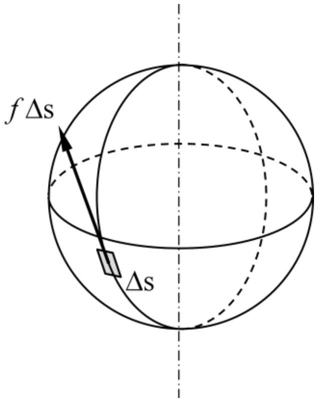

Figure 2: The air resistant force on the surface of a water droplet

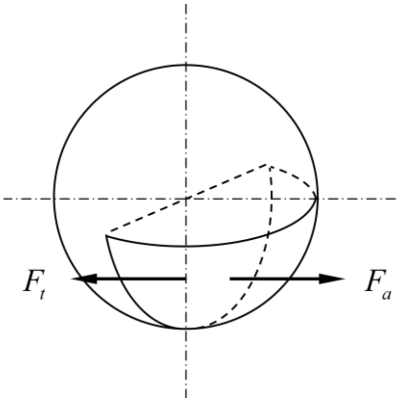

Figure 3: The horizontal component F_a of the air resistant force and the horizontal component F_t of the surface tension force acting on a quarter of the sphere in the lower part of the water droplet.

1.C. Calculate the horizontal component F_t of the surface tension force acting on a quarter of the sphere in the lower part of the water droplet as shown in Fig. 3. Express your result in terms of the surface tension coefficient of water σ and D .

1.D. When $F_t \gg F_a$ the water droplets are spherical. By using $F_t \geq 100F_a$ as the condition for spherical water droplets, find the maximal diameter of a spherical water droplet D_M . ($\sigma = 7.5 \times 10^{-2}\,\mathrm{N/m}$, $\rho = 1.0 \times 10^3\,\mathrm{kg/m^3}$, $g = 9.8\,\mathrm{m/s^2}$)

Part 2. Refraction and reflection of light rays in a spherical water droplet

2.A. Consider a light ray that is refracted into the water droplet, then reflected back once in the water droplet, and finally refracted into air, as shown in Fig. 4. α is the central angle of the incident point. Find the angle θ between the reflected ray and the reverse direction of the incident ray. Express your result in terms of α and the refraction index of water n.

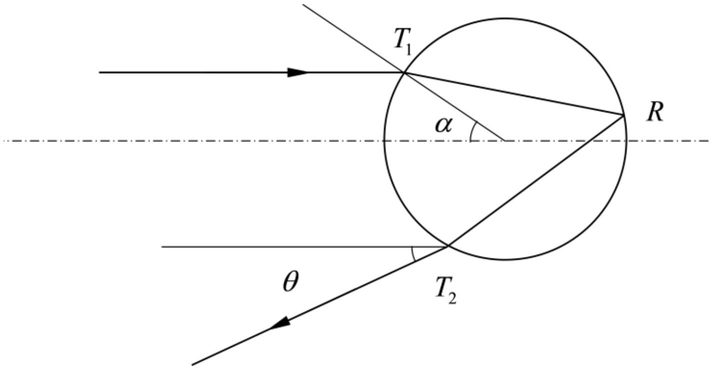

Figure 4: Refraction and reflection of light rays in a spherical water droplet

2.B. Suppose that the incident light is a parallel light with an optical intensity (optical power per unit section) equal to I_0. Find the angular optical power distribution of the reflected light $J(\theta) = \lim\limits_{\substack{\Delta\theta\to 0a \\ \Delta\phi\to 0}} \dfrac{\Delta P}{\Delta\theta\Delta\phi}$, where ΔP is the optical power within a small angular range $\Delta\phi\Delta\theta$ around a given direction (Fig. 5). Express your result in terms of α, n, I_0, the diameter of the water droplet D, the transmittance from air to water T_1, the transmittance from water to air T_2, the reflectivity in the water droplet R.

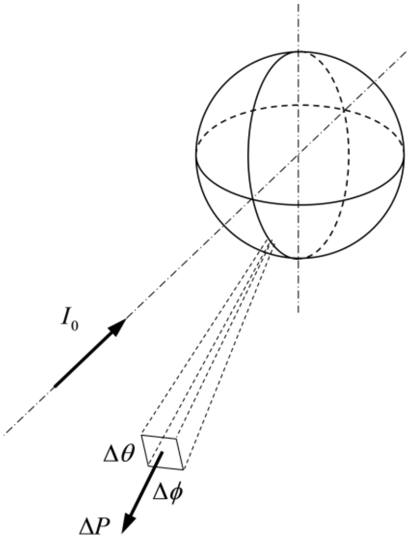

Figure 5: The angular power distribution of the reflected light.

2.C. For monochromatic lights with the wavelength $\lambda = 550\,\text{nm}$, calculate the angle θ_M at which the maxim for $J(\theta)$ occurs, and the maxim value $J(\theta_M)$ of $J(\theta)$. The waters refraction index at $\lambda = 550\,\text{nm}$ is $n_g = 1.3342$.

2.D. Suppose that the incident light is white light that contains all optical waves with the wavelength within the range $390\,\text{nm} - 780\,\text{nm}$, with equal intensities. Sketch the Intensity of the spectrum of the sunlight as a function of wavelength, reflected along the direction $\theta = \theta_M$, and indicate special values on the curve.

Part 3. Basic characteristics of the rainbow

3.A. The electromagnetic waves with the wavelength between $390\,\text{nm} - 780\,\text{nm}$ are visible lights. The waters refraction index is $n_v = 1.3439$ at $\lambda = 390\,\text{nm}$, and $n_r = 1.3316$ at $\lambda = 780\,\text{nm}$. The angular diameter of the sun is $\delta = 0.5^\circ$. Calculate the angular radius θ_0 and the angular width $\Delta\theta$ of the rainbow. Assuming the sun is right behind the observer.

3.B. The reflected sunlight is also diffracted by water droplets. The diffraction pattern resulting from a water droplet is same as the diffraction pattern resulting from a circular aperture with the same diameter as the droplet. If the angular radius of the diffraction pattern resulting from a water droplet is larger than the angular width of the rainbow, then the sunlight reflected from this water droplet does not contribute to the formation of the rainbow. Calculate the diameter d_m of the smallest water droplet that may contribute to rainbow formation.

3.C. If the altitude of the bottom of clouds was 800 m during the rain, calculate the maxim time T_M after the rain stopped, the water droplets that can contribute to rainbow formation may still be found at altitudes higher than 200 m. (The viscosity of the air is $\eta = 1.8 \times 10^{-5}$ Pa.s).

3.D. One afternoon, the rainbow appeared in certain location. If the day length was 14 hours for that day, calculate the maxim time T_r (in hours) in which the rainbow could be seen in that afternoon.

Useful formula:

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$