WoPhO Selection Round Problem 10
Tides

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In the followings, $A \simeq B$ will mean that A and B differ only in a constant (that is ignorable in this problem).

1. A. It is obvious from the definition of gravitational potential that

$$\varphi_E(r) = -G\frac{M_E}{r}.$$

The distance between the CM of the Moon and the examined point is $d = \sqrt{r^2 + R_{EM}^2 - 2rR_{EM}\cos\theta}$ according to the law of cosines. Knowing this, the gravitational potential due to the Moon is

$$\varphi_M(r,\theta) = -G\frac{M_M}{\sqrt{R_{EM}^2 + r^2 - 2R_{EM}r\cos\theta}}.$$

B. We need to use the first-order approximation for the first run only in the potential due to the Moon: ignoring differences in terms of second order in r here (as $r \approx R_E \ll R_{EM}$) gives $\sqrt{R_{EM}^2 + r^2 - 2R_{EM}r\cos\theta} = R_{EM} - r\cos\theta$; using this in the expression of the potential gives

$$\varphi_M(r,\theta) = -\frac{GM_M}{R_{EM} - r\cos\theta} = -\frac{GM_M}{R_{EM}}\left(1 + \frac{r\cos\theta}{R_{EM}}\right) \simeq -\frac{GM_M}{R_{EM}^2}r\cos\theta;$$

consequently, the resulting potential is

$$\varphi(r,\theta) = -\frac{GM_E}{r} - \frac{GM_M}{R_{EM}^2}r\cos\theta.$$

C. At this point, we should use that the water level is near the surface of the Earth, that is, $r \approx R_E$: we can express this by introducing $h := r - R_E$ and assuming that $h \ll R_E$. If we use this and the given first-order approximation in the expression of the potential we shall get

$$\varphi(r,\theta) = -\frac{GM_E}{R_E}\left(1 - \frac{h}{R_E}\right) - \frac{GM_M}{R_{EM}^2}(R_E + h)\cos\theta \simeq -\frac{GM_M R_E}{R_{EM}^2}\cos\theta + h\left(\frac{GM_E}{R_E^2} - \frac{GM_M}{R_{EM}^2}\cos\theta\right).$$

As moving the water level up or down some meters doesn't really alter its form, we can choose it conveniently. An appropriate choice is to say $h := 0$ at $\theta = \pi/2$: then $\cos\theta = 0$, so the value of the potential (in this normalization) is 0. Then the condition for equipotentiality:

$$-\frac{GM_M R_E}{R_{EM}^2}\cos\theta + h\left(\frac{GM_E}{R_E^2} - \frac{GM_M}{R_{EM}^2}\cos\theta\right) = 0$$

$$h\left(\frac{GM_E}{R_E^2} - \frac{GM_M}{R_{EM}^2}\cos\theta\right) = \frac{GM_M R_E}{R_{EM}^2}\cos\theta$$

$$h = \frac{\frac{GM_M R_E}{R_{EM}^2}\cos\theta}{\frac{GM_E}{R_E^2} - \frac{GM_M}{R_{EM}^2}\cos\theta} = \frac{\frac{M_M R_E}{R_{EM}^2}\cos\theta}{\frac{M_E}{R_E^2} - \frac{M_M}{R_{EM}^2}\cos\theta} \approx \frac{M_M R_E^3}{M_E R_{EM}^2}\cos\theta.$$

At the last approximation we have used that $M_M \ll M_E$ and $R_E \ll R_{EM}$, so the second term in the difference in the denominator is ignorable compared to the first. From this form of the expression one can see trivially that the lowest water level corresponds to $\cos\theta = -1$ ($\theta = \pi$, that is, the point opposed to the Moon), while the highest level corresponds to the point facing the Moon. The equation of this kind gives approximately (as the water level varies only a few) an ellipse: the sketch of the curve (with oversized tides) can be seen in Fig. 1.

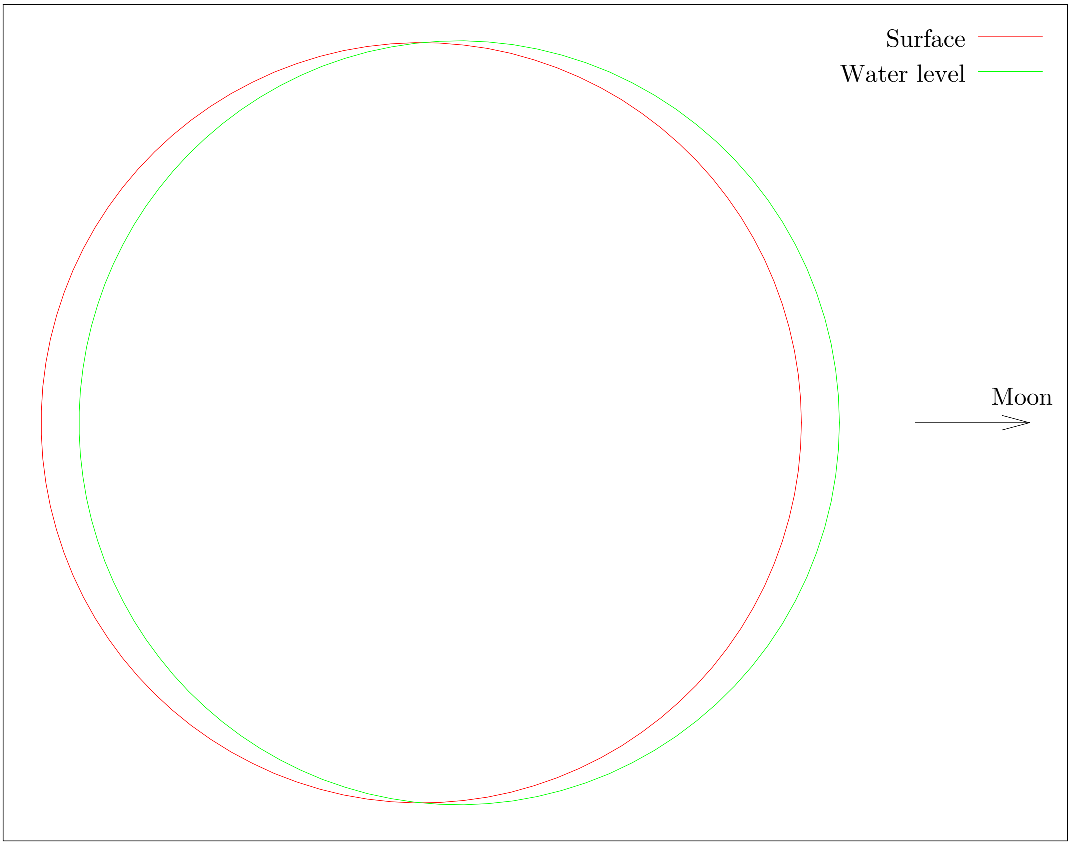

Fig. 1. The sketch of the water level in the simple model (the height of the tides is far larger than in reality for better visibility).

D. The minimum water level occurs at $\cos\theta = -1$, while the maximum is to be found at $\cos\theta = 1$ (this can be easily seen from the expression before): these levels are $h_- = -\frac{M_M R_E^3}{M_E R_{EM}^2}$ and $h_+ = \frac{M_M R_E^3}{M_E R_{EM}^2}$, respectively. The tidal height is the half of their difference:

$$\Delta h = \frac{h_+ - h_-}{2} = \frac{M_M R_E^3}{M_E R_{EM}^2} = 22.4 \text{ m}.$$

2. A. It is well-known that in a two-body system the distance to the barycentre is inversely proportional to the mass of the body. This implies that $d = R_{EM}\frac{M_M}{M_E+M_M}$. To find the angular velocity we should consider that the gravitational force between Earth and Moon should hold the Earth on its circular orbit of radius d around the barycentre. That is,

$$G\frac{M_E M_M}{R_{EM}^2} = M_E d\omega^2 = M_E \cdot R_{EM}\frac{M_M}{M_E+M_M}\omega^2$$

$$\omega = \sqrt{\frac{G(M_E+M_M)}{R_{EM}^3}}.$$

B. The centrifugal force per unit mass in a distance x from the axis is $x\omega^2$ and points away from it. This is analogous to the force exerted by a string ($F = -kx$) that is proportional to the distance: however, this force points towards the „axis". The potential energy of a spring is given by $\frac{1}{2}kx^2$ as it is well-known: our analogy implies then that the centrifugal potential is $\varphi_C = -\frac{1}{2}x^2\omega^2$.

C. From the law of cosines $x^2 = d^2 + r^2 - 2rd\cos\theta$. Using this and the expression of ω, the centrifugal potential is

$$\varphi_C(r,\theta) = -\frac{G(M_E+M_M)}{2R_{EM}^3}\left(\frac{R_{EM}^2 M_M^2}{(M_E+M_M)^2} - \frac{2R_{EM}M_M}{M_E+M_M}r\cos\theta + r^2\right) \simeq$$

$$\simeq -\frac{G(M_E+M_M)r^2}{2R_{EM}^3} + \frac{GM_M r\cos\theta}{R_{EM}^2}$$

D. The given approximation implies that one should calculate to the second order in r in $\varphi_M(r,\theta)$:

$$\varphi_M(r,\theta) = -\frac{GM_M}{\sqrt{R_{EM}^2 - 2R_{EM} r\cos\theta + r^2}} = -\frac{GM_M}{R_{EM}}\left(1 - \frac{2r\cos\theta}{R_{EM}} + \frac{r^2}{R_{EM}^2}\right)^{-1/2} =$$

$$= -\frac{GM_M}{R_{EM}}\left(1 + \frac{r\cos\theta}{R_{EM}} - \frac{r^2}{2R_{EM}^2} + \frac{3}{8}\left(\frac{2r\cos\theta}{R_{EM}}\right)^2\right) \simeq$$

$$\simeq -\frac{GM_M r\cos\theta}{R_{EM}^2} + \frac{GM_M r^2}{2R_{EM}^3}(1 - 3\cos^2\theta).$$

Using other terms of the potential that were found so far gives the total potential as

$$\varphi(r,\theta) = -\frac{GM_E}{r} - \frac{GM_M r\cos\theta}{R_{EM}^2} + \frac{GM_M r^2}{2R_{EM}^3}(1 - 3\cos^2\theta) - \frac{G(M_E + M_M)r^2}{2R_{EM}^3} + \frac{GM_M r\cos\theta}{R_{EM}^2} =$$

$$= -\frac{GM_E}{r} - \frac{Gr^2}{2R_{EM}^3}(M_E + 3M_M\cos^2\theta).$$

E. We're going to use the substitution $r := R_E + h$ and first-order approximation again. This gives now

$$\varphi(r,\theta) = -\frac{GM_E}{R_E} + \frac{GM_E}{R_E^2}h - \frac{G(R_E^2 + 2R_E h)}{2R_{EM}^3}(M_E + 3M_M\cos^2\theta) \simeq$$

$$\simeq -\frac{3GR_E^2 M_M}{2R_{EM}^2}\cos^2\theta + h\left(\frac{GM_E}{R_E^2} - \frac{GR_E(M_E + 3M_M\cos^2\theta)}{R_{EM}^3}\right)$$

We can choose the place of the water level conveniently: let $h := 0$ at $\theta = \pi/2$; then the resulting potential is 0 again, the condition of equipotentiality:

$$0 = -\frac{3GR_E^2 M_M}{2R_{EM}^2}\cos^2\theta + h\left(\frac{GM_E}{R_E^2} - \frac{GR_E(M_E + 3M_M\cos^2\theta)}{R_{EM}^3}\right)$$

$$h\left(\frac{GM_E}{R_E^2} - \frac{GR_E(M_E + 3M_M\cos^2\theta)}{R_{EM}^3}\right) = \frac{3GR_E^2 M_M}{2R_{EM}^2}\cos^2\theta$$

$$h = \frac{\frac{3GR_E^2 M_M}{2R_{EM}^2}\cos^2\theta}{\frac{GM_E}{R_E^2} - \frac{GR_E(M_E+3M_M\cos^2\theta)}{R_{EM}^3}} = \frac{\frac{3R_E^2 M_M}{2R_{EM}^2}\cos^2\theta}{\frac{M_E}{R_E^2} - \frac{R_E(M_E+3M_M\cos^2\theta)}{R_{EM}^3}} \approx \frac{3R_E^4 M_M}{2R_{EM}^3 M_E}\cos^2\theta$$

The last approximation uses that the second term in the difference of the denominator is ignorable compared to the first one (they are in the order of magnitude of 10^{11} and 10^6 SI-units, respectively). It is easy to see from this expression that the lowest water level corresponds to $\cos^2\theta = 0$ (that is, $\theta = \pi/2$), while the highest level corresponds to $\cos^2\theta = 1$ ($\theta = 0$ or $\theta = \pi$). The sketch of the water level (with oversized tides) can be seen in Fig. 2.

F. The lowest water level is at $\theta = \pi/2$, here $h_- = 0$ as one can easily see; the maximal water level at $\theta = 0$ is $h_+ = \frac{3R_E^4 M_M}{2R_{EM}^3 M_E}$. The tidal height is then:

$$\Delta h = \frac{h_+ - h_-}{2} = \frac{3R_E^4 M_M}{4R_{EM}^3 M_E} = 28.3 \text{ cm}.$$

G. What we have to do is simply changing indices M to S, that is, the water level height is

$$h_S = \frac{\frac{3R_E^2 M_S}{2R_{ES}^2}\cos^2\theta}{\frac{M_E}{R_E^2} - \frac{R_E(M_E+3M_S\cos^2\theta)}{R_{ES}^3}} \approx \frac{3R_E^4 M_S}{2R_{ES}^3 M_E}\cos^2\theta.$$

(We should note that the second term is ignorable in this case, too: it's in the order of magnitude of 10^4 SI-units compared to that of 10^{11} SI-units of the first term.) Since we can use this approximation, the tidal height caused by the Sun is given by

$$\Delta h_S = \frac{3R_E^4 M_S}{4R_{ES}^3 M_E} = 12.4 \text{ cm}.$$

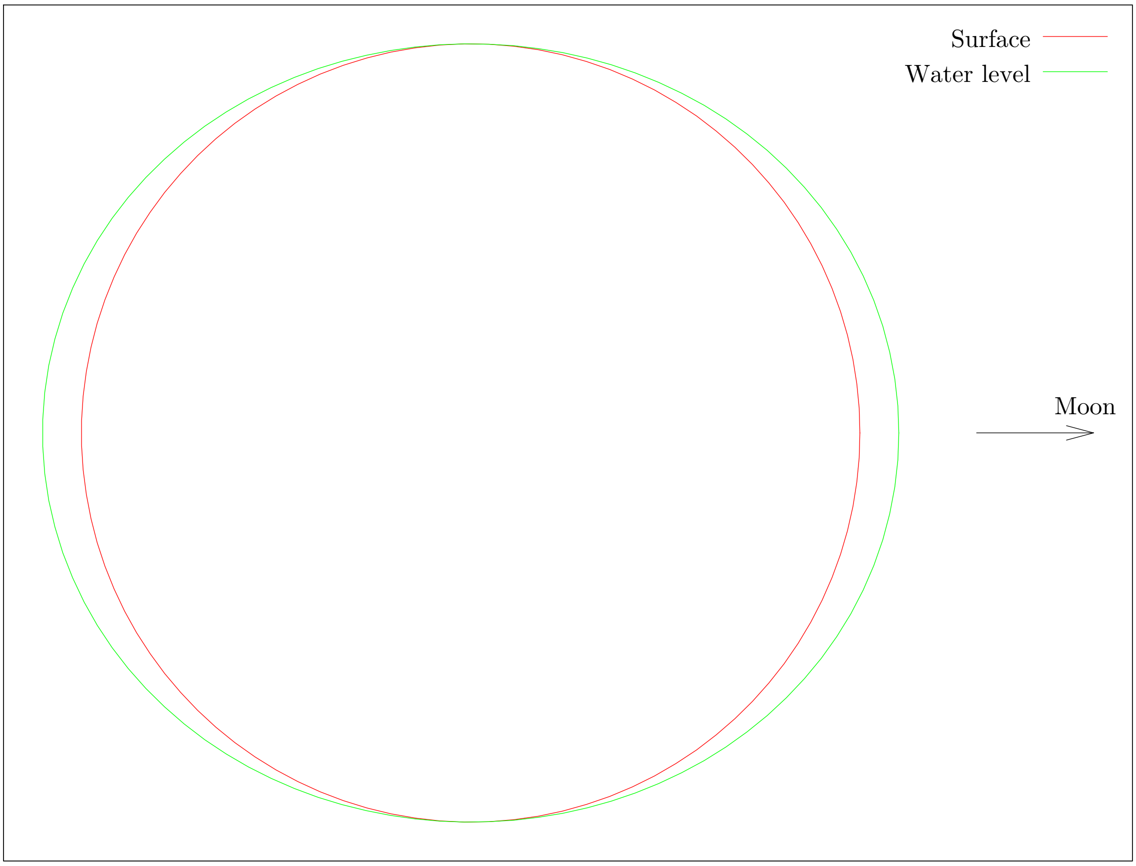

Fig. 2. The sketch of the water level in the sophisticated model (the height of the tides is far larger than in reality for better visibility).