Magnetic monopole

A problem proposed for the problem competition of WoPhO 2012

21 June 2012

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Magnetic monopole

Introduction

Our everyday experience shows that the magnetic poles always exist in pairs – North and Sought. Breaking a magnet results in the appearance of a new pair of opposite magnetic poles on the two broken ends. The fundamental laws of physics, however, do not contradict the existence of magnetically charged particles called *magnetic monopoles*. The magnetic monopole is an object possessing just one magnetic pole, either North, or Sought, which are analogues of the positive and negative electric charges respectively. Thus, the magnetic field of the monopole is similar to the electric field created by a static electric charge, i.e. its force lines begin or end at the point where the monopole is located. This property is in contrast to the closed force lines of the magnetic field created by permanent magnets (magnetic dipoles) and electric currents. The concept of magnetic monopole was introduced in 1932 by the famous physicist Paul Dirac. On the basis of quantum mechanics he proved that the existence of magnetic monopoles can explain the existence of the elementary electric charge. That is why the physicists do not cease their efforts to discover magnetic monopoles experimentally.

In the following questions you are going to establish some properties of the magnetic monopoles by analyzing simple model situations (though experiments). You may assume that all laws of physics known to you apply to the magnetic monopoles, except the statement for closed force lines of the magnetic field. The velocities considered in this problem are much smaller than the speed of light and, therefore, you may neglect the relativistic effects on time, length and mass.

Use the following physical constants in your solution:

magnetic permeability of vacuum: $\mu_0 = 4\pi\times10^{-7}\,\mathrm{H/m}$;

electric permittivity of vacuum: $\varepsilon_0 = 8.854\times10^{-12}\,\mathrm{F/m}$;

speed of light: $c = 2.998\times10^{8}\,\mathrm{m/s}$;

elementary electric charge: $e = 1.602\times10^{-19}\,\mathrm{C}$;

Plank’s constant: $h = 6.626\times10^{-34}\,\mathrm{J\cdot s}$.

Questions

Question 1. When exposed to external magnetic field of induction **B**, a monopole of magnetic charge q_m experiences a force:

$$\mathbf{F} = q_m \mathbf{B}$$

a. Derive the unit of magnetic charge in terms of the basic SI units: kilogram, meter, second, ampere.

Question 2. Electric current I circulates along a circular loop of radius a. A monopole of magnetic charge q_m is situated on the axis of the loop at a point of coordinate z relative to its center, as shown in Figure 1. The positive direction of the axis Z and the direction of current circulation are related through the right-hand rule.

a. Find out an expression for the z-component F_z of the force acting on the monopole.

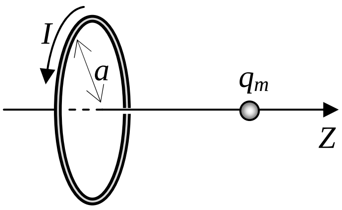
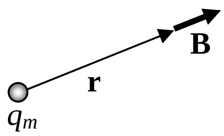


Figure 1.

Question 3. When at rest, the magnetic monopole creates static magnetic field, similar to the electric field produced by a static electric charge. The magnetic induction **B** at a point of position-vector **r**, relative to the monopole (see Figure 2), is given by the equation:

$$\mathbf{B} = \frac{k_m q_m \mathbf{r}}{r^3}$$



where k_m is a coefficient of proportionality and $r = |\mathbf{r}|$.

Figure 2.

a. By analyzing the system described in Question 2, express the unknown coefficient k_m through the provided fundamental constants.

b. Formulate by means of equation the Gauss law for the flux Φ of the magnetic induction created by the magnetic monopole.

Question 4. A moving electric charge creates magnetic field. Likewise, the moving magnetic monopole produces electric field with circular force-lines (i.e. a vortex field) concentric with the direction of motion of the monopole (see Figure 3). Consider a monopole of magnetic charge q_m moving along a straight line with a constant velocity **v**.

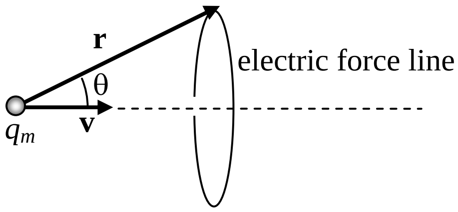


Figure 3.

a. Derive an expression for the intensity **E** of the electric field created by the monopole in a point of position-vector **r** making an angle θ with the vector of velocity, as shown in Figure 3. Use vector notations in your final answer in order to specify both, the magnitude, and the direction of the electric field.

b. Suppose that a positive electric charge q_e and a positive magnetic charge q_m are moving against you, perpendicularly to the sheet of paper. Draw two arbitrary force lines in your answer sheet – one for the magnetic field created by the electric charge and another, for the electric field created by the magnetic monopole. <u>Indicate the directions of the two lines.</u>

Question 5. The analogy between electric and magnetic charges is found also in the way they interact with external magnetic and electric fields respectively. Similarly to the Lorentz force acting on an electric charge moving in magnetic field, the magnetic charge experiences a force when it moves in electric field.

a. Propose and analyze a thought experiment in order to derive an expression for the "Lorenz" force acting on a monopole of magnetic charge q_m moving with a velocity **v** in electric field of intensity **E**. Use vector notations in your final answer in order to specify both, the magnitude and the direction of the force. When describing your thought experiment, use proper drawings and short comments to them instead of a lengthy text.

Question 6. A point particle of electric charge q_e is confined to move along a circle without any resistance or friction, as shown in Figure 4. A monopole of magnetic charge q_m passes through the plane of the circle by moving along its axis Z from $z \to -\infty$ to $z \to +\infty$.

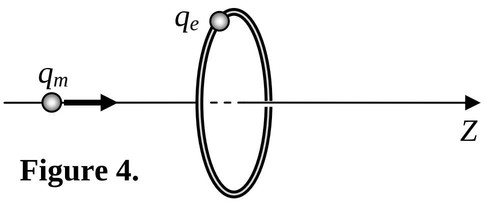


Figure 4.

a. Obtain an expression for the change in Z–component, ΔL_z, of the angular momentum of the electrically charged particle during the whole motion of the magnetic monopole. Express your answer in terms of q_e, q_m and fundamental constants only.

Question 7. In his famous work on magnetic monopoles Paul Dirac has argued that if just one single magnetic monopole existed in the Universe, all electric charges should be multiple of a specific elementary electric charge, whose magnitude is related to the magnetic charge of that monopole. Historically, it is the first hypothesis in physics, which explains the existence of the elementary electric charge.

Consider the system described in Question 6, assuming that all magnetic monopoles existing in the Nature have magnetic charges of the same magnitude, $+q_m$ and $-q_m$ respectively .

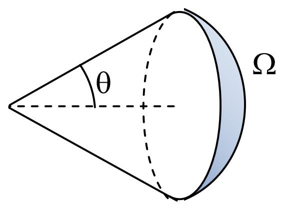
a. By applying the concepts of quantum physics to the motion of electrically charged particle along the circular orbit, derive a relationship between the elementary electric charge e, assumed to be the charge of the electron, and the magnetic charge q_m of the monopole. Calculate q_m numerically.

b. The electron possesses a self magnetic moment of $p_m = 9.274 \times 10^{-24}\,\mathrm{A.m^2}$. By assuming that the magnetic properties of the electron are due to a pair of spatially separated point magnetic monopoles of opposite magnetic charges, $+q_m$ and $-q_m$ respectively; calculate the distance d between these monopoles.

Useful math:

- The solid angle Ω enclosed by a cone of half-opening angle θ (see the figure) is:

$$\Omega = 2\pi(1-\cos(\theta))$$



- Depending on your approach to the solution you may need the following integral:

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{(z^2+a^2)^{3/2}} = \frac{2}{a^2}$$

Solution

Question 1.

a. The simplest approach is to compare the expression

$$\mathbf{F} = q_m \mathbf{B}$$

for the force acting on the monopole with the Ampere's formula for the magnetic force acting on an element **d** of a conductor carrying current *I*:

$$\mathbf{F} = I\mathbf{d} \times \mathbf{B}$$

The equivalence of the dimensionality of the right-hand sides of the two equations shows that:

$[q_m] = \mathrm{A} \cdot \mathrm{m}$ (ampere × meter)

Alternatively, one may use the expression for the Lorenz force applied on a moving electric charge;

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

which gives:

$$[q_m] = \mathrm{C} \cdot \mathrm{m} \cdot \mathrm{s}^{-1} = \mathrm{A} \cdot \mathrm{m}$$

since $\mathrm{A} = \mathrm{C} \cdot \mathrm{s}^{-1}$.

Question 2.

a. The magnetic field of the current is found by means of the Biot–Savart law. The magnetic field d*B* created by a small current element of length d*s* is:

$$dB = \frac{\mu_0 I ds}{4\pi r^2}$$ (See Figure S1 for the notations).

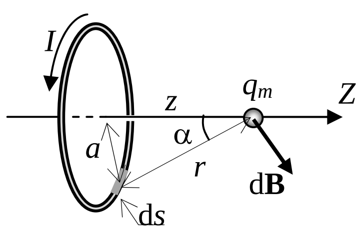


Figure S1.

The projection of the magnetic induction on the axis *Z* is:

$$dB_z = dB \sin\alpha = \frac{\mu_0 I a ds}{4\pi\left(a^2 + z^2\right)^{3/2}}$$

since $\sin\alpha = a/r$ and $r = \left(a^2 + z^2\right)^{1/2}$. We obtain the total magnetic field at a point *z* by integrating the above expression along the circle:

$$B_z = \frac{\mu_0 I a^2}{2\left(a^2 + z^2\right)^{3/2}}$$

and find respectively the force acting on the monopole:

$$\boxed{F_z = \frac{\mu_0 q_m I a^2}{2\left(a^2 + z^2\right)^{3/2}}}$$

Question 3.

a. The monopole, on its turn, exercises a force **F′** on the current loop. That force can be expressed by considering first the Ampere's force applied on a small element d*s* of the loop:

$$\mathrm{d}F_z' = -(IB\mathrm{d}s)\sin\alpha = -\frac{k_m q_m I \mathrm{d}s}{r^2}\sin\alpha = -\frac{k_m q_m I a \mathrm{d}s}{\left(a^2 + z^2\right)^{3/2}}.$$

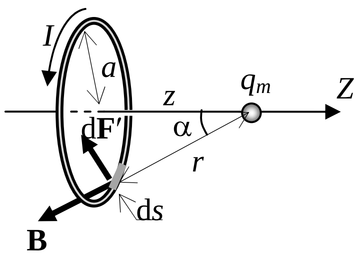


Figure S2.

After integration along the circle we obtain:

$$F_z' = -\frac{2\pi k_m q_m I a^2}{\left(a^2 + z^2\right)^{3/2}}.$$

According to the Newton's third law:

$$F_z' = -F_z$$

This condition is satisfied provided that:

$$\boxed{k_m = \frac{\mu_0}{4\pi}}$$

b. Now, consider a spherical surface of radius *r* surrounding the magnetic monopole. The flux of magnetic induction through that surface is:

$$\boxed{\Phi = B(r) \times (4\pi r^2) = \mu_0 q_m}$$

That last relation is a formulation of the Gauss law in the case of magnetic monopoles.

Question 4.

a. The question could be addressed differently depending on the choice of the frame of reference.

I approach Laboratory reference system – the monopole moves with velocity **v** along *Z*–axis.

Consider the circular force-line of the electric field passing through the point of interest (see Figure S3). The radius of the force line is related to the instantaneous values of *r* and θ as $a = r\sin\theta$. The origin of the coordinate system is chosen at the center of the circle. The flux of the magnetic field through the plane of the force line is:

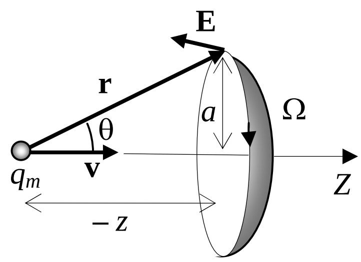


Figure S3

$$\Phi = \mu_0 q_m \frac{\Omega}{4\pi}$$

where

$$\Omega = 2\pi(1-\cos\theta)$$

is the solid angle enclosed by the conical surface of half-aperture θ. Taking into account that $\cos\theta = -z/(z^2+a^2)^{1/2}$ we obtain:

$$\Phi = \frac{\mu_0 q_m}{2}\left(1 + z/(z^2+a^2)^{1/2}\right)$$

The EMF, induced along the force-line, is:

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mu_0 q_m a^2}{2(z^2+a^2)^{3/2}}\,v$$

where $v = \mathrm{d}z/\mathrm{d}t$ is the speed of the monopole. The minus sign indicates that the EMF acts against the circulation direction defined by the right-hand rule, as shown in Figure S3.

The EMF is related to the tangential component *E* of the electric-filed by:

$$\mathcal{E} = E(2\pi a)$$

Therefore:

$$E = -\frac{\mu_0 q_m a}{4\pi(z^2+a^2)^{3/2}}\,v = -\frac{\mu_0 q_m v\sin\theta}{4\pi r^2}$$

The "–" sign indicates that the direction of the vector **E** is against the positive direction of the force line, as shown in Figure S3. Therefore, in a vector form, the electric field created by the moving monopole, is:

$$\boxed{\mathbf{E} = -\frac{\mu_0 q_m \mathbf{v}\times\hat{\mathbf{r}}}{4\pi r^2} = -\frac{\mu_0 q_m \mathbf{v}\times\mathbf{r}}{4\pi r^3}}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit-vector along the position-vector **r**.

<u>II approach</u> A “shortcut” solution – the rest-system of the monopole

Consider a stationary point electric charge q_e situated at the position defined by the vector **r** relative to the monopole (see Figure S4). In the monopole rest system the electric charge moves with a velocity –**v**, against the positive direction of axis *Z*. Since only static magnetic field is present at that system (the monopole is at rest) the electric charge experiences a Lorenz force:

$$\mathbf{F} = q_e(-\mathbf{v})\times\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 q_e q_m \mathbf{v}\times\mathbf{r}}{4\pi r^3}$$

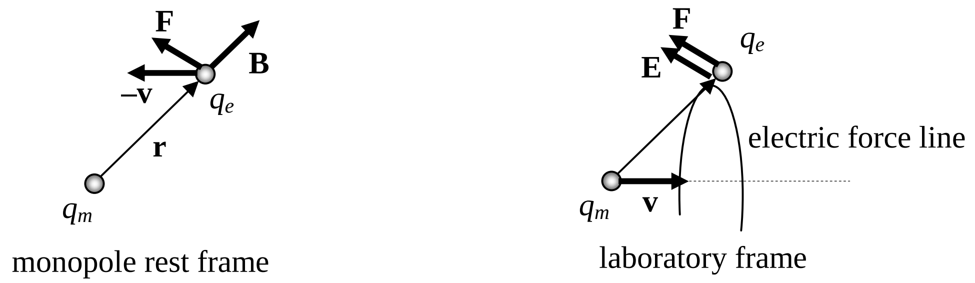


Figure S4

In a non-relativistic approximation, the same force is sensed by the electric charge in the laboratory reference system. In that system, however, it is due to the presence of an electric field at the point where the electric charge is situated. By following the definition of the intensity of electric field, we obtain:

$$\boxed{\mathbf{E} = \frac{\mathbf{F}}{q_e} = -\frac{\mu_0 q_m \mathbf{v}\times\mathbf{r}}{4\pi r^3}}$$

<u>III approach</u> An energetic balance approach

Suppose that the magnetic monopole moves along the axis of a circular current-carrying loop, similar to that shown in Figure S1. According to the result of Question 2, the monopole experiences a force:

$$F_z = \frac{\mu_0 q_m I a^2}{2\left(a^2+z^2\right)^{3/2}}$$

Since the monopole + current system is closed, the total instantaneous power of all forces in the system is zero:

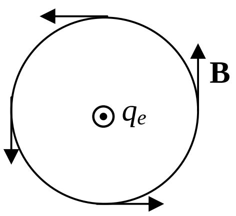
$$F_z v + \mathcal{E} I = 0$$

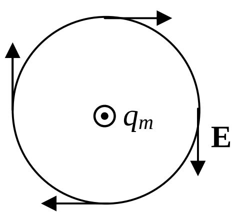
where $\mathcal{E}$ is the EMF induced in the loop. Thus, we obtain:

$$\mathcal{E} = -\frac{\mu_0 q_m a^2}{2(z^2 + a^2)^{3/2}} v$$

After that we recover all the subsequent steps described in the first approach in order to find **E**.

b. The two drawings below indicate the directions of the force lines of the magnetic and electric fields created by moving positive electric and magnetic charges respectively.





Question 5.

a. A simple model system is shown in Figure S5. A static electric charge q_e is located at the origin of the coordinate system. A magnetic monopole q_m with instantaneous position-vector **r** moves with velocity **v**. The electric field, felt by the monopole, is:

$$\mathbf{E} = \frac{q_e \mathbf{r}}{4\pi\varepsilon_0 r^3}$$

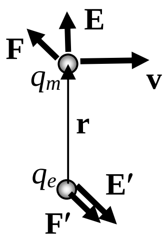


Figure S5

On its turn, the moving magnetic monopole creates electric field at the origin of the coordinates system:

$$\mathbf{E}' = -\frac{\mu_0 q_m \mathbf{v} \times (-\mathbf{r})}{4\pi r^3} = \frac{\mu_0 q_m \mathbf{v} \times \mathbf{r}}{4\pi r^3}$$

Therefore, the electric charge experiences an electric force:

$$\mathbf{F}' = q_e \mathbf{E}' = \frac{\mu_0 q_m q_e \mathbf{v} \times \mathbf{r}}{4\pi r^3}$$

According to the Newton's third law the "Lorenz" force, applied on the monopole, is:

$$\mathbf{F} = -\mathbf{F}' = -\frac{\mu_0 q_m q_e \mathbf{v}\times\mathbf{r}}{4\pi r^3} = -\mu_0\varepsilon_0 q_m \mathbf{v}\times\mathbf{E}$$

or, taking into account that $\mu_0\varepsilon_0 = 1/c^2$:

$$\boxed{\mathbf{F} = -\frac{1}{c^2} q_m \mathbf{v}\times\mathbf{E}}$$

Question 6. The rate of change of the angular momentum of the electric charge is equal to the torque of the electric force acting on it:

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = Fa = q_e aE$$

Here E is the tangential component of the electric field created by the moving monopole. By using the result derived in Question 4, we have:

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = -\frac{\mu_0 q_m q_e a^2}{4\pi(z^2+a^2)^{3/2}}\upsilon .$$

Since the displacement of the monopole for a time interval dt is:

$$\mathrm{d}z = \upsilon \mathrm{d}t$$

we can obtain the change of angular momentum by integration over z:

$$\boxed{\Delta L_z = -\int_{-\infty}^{+\infty}\frac{\mu_0 q_m q_e a^2}{4\pi(z^2+a^2)^{3/2}}\mathrm{d}z = -\frac{\mu_0 q_m q_e}{2\pi}}.$$

The integration could be avoided by noticing that the torque of the electric force is proportional to the EMF induced in the circular loop:

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = q_e aE = q_e\frac{\mathcal{E}}{2\pi} .$$

From the Faraday's law: $\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$, it follows that:

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = -\frac{q_e}{2\pi}\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

where Φ is the flux of the magnetic field through the plane of the loop. Therefore, the change of the angular momentum of the electric charge for arbitrary time interval is proportional to the change of the magnetic flux through the loop for the same interval:

$$\Delta L_z = -\frac{q_e}{2\pi}\Delta\Phi$$

According to the formula: $\Phi = \frac{\mu_0 q_m}{2}\left(1 + z/(z^2 + a^2)^{1/2}\right)$, derived in Question 4, the magnetic flux through the circle changes from 0 when $z \to -\infty$ to $\mu_0 q_m$ as $z \to +\infty$. Thus, we obtain the same result:

$$\boxed{\Delta L_z = -\frac{\mu_0 q_m q_e}{2\pi}}$$

Question 7.

a. According to the Bohr's quantization rule, the change of the angular momentum along a specified axis is a multiple of the reduced Plank's constant:

$$\Delta L_z = n\hbar = n\frac{h}{2\pi}$$

By utilizing the expression for ΔL_z, derived in the last question, we obtain:

$$\frac{\mu_0 q_m q_e}{2\pi} = \frac{nh}{2\pi} \quad \text{or} \quad q_e = n\frac{h}{\mu_0 q_m}.$$

This result implies that the electric charge is multiple of a universal elementary electric charge $\frac{h}{\mu_0 q_m}$. If we identify the elementary electric charge with the charge of the electron, we can find the magnitude of the magnetic charge of the monopole:

$$\boxed{q_m = \frac{h}{\mu_0 e} = 3.29\times10^{-9}\,\mathrm{A\cdot m}}$$

b. If the magnetic moment of the electron is due to a pair of opposite magnetic charges then the following relation holds true:

$$p_m = q_m d \quad \text{or} \quad \boxed{d = \frac{p_m}{q_m} = 2.82\times10^{-15}\,\mathrm{m}}$$

Marking scheme

Question	Element of the solution	Points	Comments
1 a	Compares the formula for the force applied on the monopole with a relevant formula for the magnetic force applied to a conductor or a moving electric charge	0.5	
	Correct final expression for $[q_m]$	0.3	Deduce 0.1 if $C.s^{-1}$ is not converted to A
	Subtotal on Question 1 a	**0.8**	
2 a	Applies the Biot-Savart law for an infinitesimal element of the loop	0.5	
	Projects correctly dB onto axis Z	0.2	
	Correct integration (summation) along the loop	0.2	
	Final expression for Fz	0.2	
	Subtotal on Question 2 a	**1.1**	
3 a	Provides expression for the Ampere's force on an infinitesimal element of the loop	0.5	Assign 0.5 for the whole part if k_m is related to μ_0 on the basis of a dimensional analysis.
	Projects the force onto axis Z	0.2	
	Integration (summation) along the loop	0.1	
	Final expression for F'z	0.1	
	Applies the Newton's third law for Fz and F'z	0.5	
	Correct expression for k_m	0.2	
	Subtotal on Question 3 a	**1.6**	
3 b	Applies the definition of a magnetic flux for a spherical surface around the monopole	0.3	Assign the full mark if the magnetic flux is calculated by integration over arbitrary closed surface
	Correct formulation of the Gauss law	0.2	
	Subtotal on Question 3 b	**0.5**	

<table>
<tr><td rowspan="6">4 a</td><td>Calculates the magnetic flux through the area enclosed by the force line</td><td>0.5</td><td rowspan="2">Alternatively:

Formulates the energy balance equation: 0.8
and
provides expression for the induced EMF: 0.2</td><td rowspan="5">Alternatively:

Expression for the Lorenz force experienced by an electric charge in the monopole rest-frame 1.0

Uses the invariance of force and applies the definition of electric field intensity in the laboratory frame: 0.5

Correct vector expression for the electric field intensity: 0.2</td></tr>
<tr><td>Applies the Faraday's law and finds an expression for the EMF induced along the force line</td><td>0.5</td></tr>
<tr><td>Relates the induced EMF to the tangential component of the electric field</td><td>0.3</td><td>0.3</td></tr>
<tr><td>Correct expression for the tangential component E</td><td>0.2</td><td>0.2</td></tr>
<tr><td>Vector expression for the electric field intensity</td><td>0.2</td><td>0.2</td></tr>
<tr><td>Subtotal on Question 4 a</td><td>1.7</td><td colspan="2"></td></tr>
<tr><td rowspan="3">4 b</td><td>Draws a circular magnetic force line for the moving electric charge and indicates an anti-clockwise direction on it</td><td>0.2</td><td colspan="2"></td></tr>
<tr><td>Draws a circular electric force line for the moving magnetic monopole and indicates a clockwise direction on it</td><td>0.2</td><td colspan="2"></td></tr>
<tr><td>Subtotal on Question 4 b</td><td>0.4</td><td colspan="2"></td></tr>
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5 a	Clear drawing for a system composed of a moving magnetic monopole and a stationary electric charge	0.2	Follow the same marking scheme for alternative systems.
	Provides expression for the force **F** ' acting on the electric charge	0.2	
	Provides expression for the force **F** acting on the moving monopole	0.2	
	Applies the Newton's third law for **F** and **F'**	0.2	
	Provides vector expression for **F**	0.2	
	Subtotal on question 5 a	**1.0**	
6 a	Relates dL_z/dt to the torque created by the electric force applied on the electric charge	0.3	
	Provides expression for the torque on the electric charge through the instantaneous position z and velocity v of the monopole	0.3	**Alternatively:** Relates the torque to the induced EMF
	Uses the expression dz=vdt for the displacement of the monopole along z	0.2	Uses the Faraday's law and to derive relation between ΔLz and $\Delta\Phi$
	Integrates dLz over monopole position z in proper limits	0.3	Calculates $\Delta\Phi$ for the whole motion
	Final expression for ΔL_z	0.2	
	Subtotal on Question 6 a	**1.3**	
7 a	Applies the Bohr's rule for quantization of the angular momentum	0.5	**Alternatively**: States that the perimeter of the orbit is a multiple of de Broglie wavelength
	Expresses the electric charge q_e as a multiple of $h/(\mu_0 q_m)$	0.2	
	Provides correct expression for the magnetic charge of the monopole	0.2	
	Calculates q_m numerically	0.2	
	Subtotal on Question 7 a	**1.1**	

7 b	Provides an expression for the magnetic dipole moment of the electron through the charge of the monopoles and the distance between them	0.3	
	Calculates *d* numerically	0.2	
	Subtotal on Question 7 b	**0.5**	
Total score for the Problem		**10.0**	