# TORNADO

**Theoretical problem for World Physics Olympiad 2012**

*Oki Gunawan, Ph.D.*
*IBM T J Watson Research Center, Yorktown Height, NY 10598 USA*

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# SOLUTIONS :

## [Q.1] <u>The calm weather</u>

**(a)** Since it is a calm weather (no wind) we assume the atmosphere is in hydrostatic condition. Consider a column of air as shown below. The atmospheric pressure is higher at lower altitude since it needs to support more column of air above.

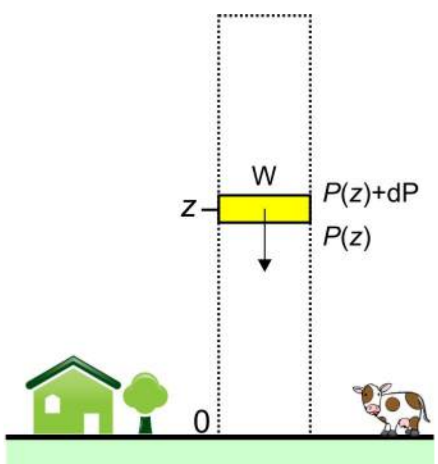


Figure 1. Diagram of an air column for pressure versus altitude calculation

At equilibrium:

$$dP = -dF/A = -\frac{dm\,g}{A} = -\rho\, g\, dz \qquad (1)$$

Using ideal gas law:

$$PV = nRT \quad \text{and} \quad P = \rho\frac{RT}{M_A} \quad \text{or} \quad \rho = P\frac{M_A}{RT} \qquad (2)$$

where $M_A$ = 0.0290 kg/mol, is the molar mass of the air. Now we have:

$$\frac{dP}{P} = -\frac{g\,M_A}{R\,T}\,dz \tag{3}$$

Assuming constant temperature $T_0$, solving this yields:

$$P(z) = P_0^{-\alpha z}\text{, with} \qquad \alpha = \frac{g\,M_A}{R\,T_0} \tag{4}$$

**(b)** Now the temperature varies with altitude: $T = T_0 - b\,z$. Eq. (3) becomes :

$$\int_{P_0}^{P} \frac{dP}{P} = -\frac{g\,M_A}{R}\int_0^z \frac{dz}{T_0 - b\,z}\text{, which gives:} \qquad \ln\frac{P}{P_0} = -\frac{g\,M_A}{b\,R}\ln\left(\frac{T_0 - b\,z}{T_0}\right) \tag{5}$$

Finally:

$$P(z) = P_0\left(1 - \frac{b\,z}{T_0}\right)^{-\frac{g\,M_A}{b\,R}} \tag{6}$$

**(c)** At point B we have:

$$P_B = 1e5\left(1 - \frac{(6.5e-3)(1e3)}{288}\right)^{-\frac{(9.8)(0.029)}{(6.5E-3)(8.314)}} = 8.87\times 10^4 \text{ Pa} \tag{7}$$

Thus the pressure is lower by 11.3% at the base of the cumulonimbus cloud compared to the ground.

## [Q.2] **The shape of the tornado**

**(a)** Because point G is in the isobar boundary layer we have: $P_G = P_B$. Using Bernoulli's equation at point A and G, and assuming constant air mass density :

$$P_A + \cancel{\tfrac{1}{2}\rho v_A^{\,2}} + \cancel{\rho g z_A} = P_G + \tfrac{1}{2}\rho\,v_G^{\,2} + \cancel{\rho g z_G} \tag{8}$$

$$v_G = \sqrt{\frac{2(P_A - P_B)}{\rho}} = \sqrt{\frac{2(1e5 - 0.887e5)}{1.2}} = 137\,\text{m/s} \tag{9}$$

which is within the range of speed for a deadly tornado ($v$ ~ 100 m/s).

## C. SOLUTION

**(b)** Due to conservation of angular momentum we have (using point G as a reference or any point P at altitude $z$) :

$$m v_C r_C = m v_G r_G = m v r \tag{10}$$

Thus:

$$v(r) = \frac{v_G r_G}{r} \tag{11}$$

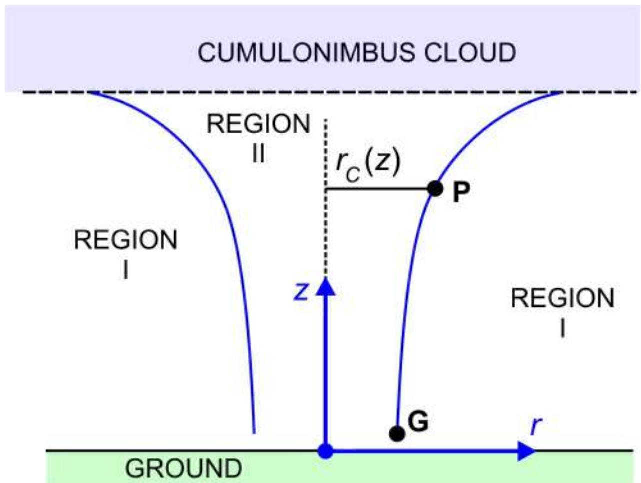


Figure 2. Tornado cross section diagram

**(c)** Consider point P along the boundary between region I and II as shown above. This boundary layer is also an isobar boundary layer thus $P_P = P_G$. We have:

$$\cancel{P_G} + \tfrac{1}{2}\rho v_G^2 + \cancel{\rho g z_G} = \cancel{P_P} + \tfrac{1}{2}\rho v_P^2 + \rho g z \tag{12}$$

$$v_G^2 = v_P^2 + 2 g z \tag{13}$$

From part (b) we have: $v(r) = v_G r_G / r$. Note that $v$ is only a function of $r$ not altitude $z$ as stated in the problem. Thus we have:

$$z = \frac{v_G^2}{2g}\left[1 - \left(\frac{r_G}{r}\right)^2\right] \tag{14}$$

Alternatively we can express the tornado shape as a function of radius versus altitude below and plot it in Figure 3 below.

$$\frac{r}{r_G} = \frac{1}{\sqrt{1 - \frac{2gz}{v_G^2}}} \tag{15}$$

As we can see, in comparison with a real life tornado picture, this equation describe the general feature of tornado shape remarkably well despite a very simple model.

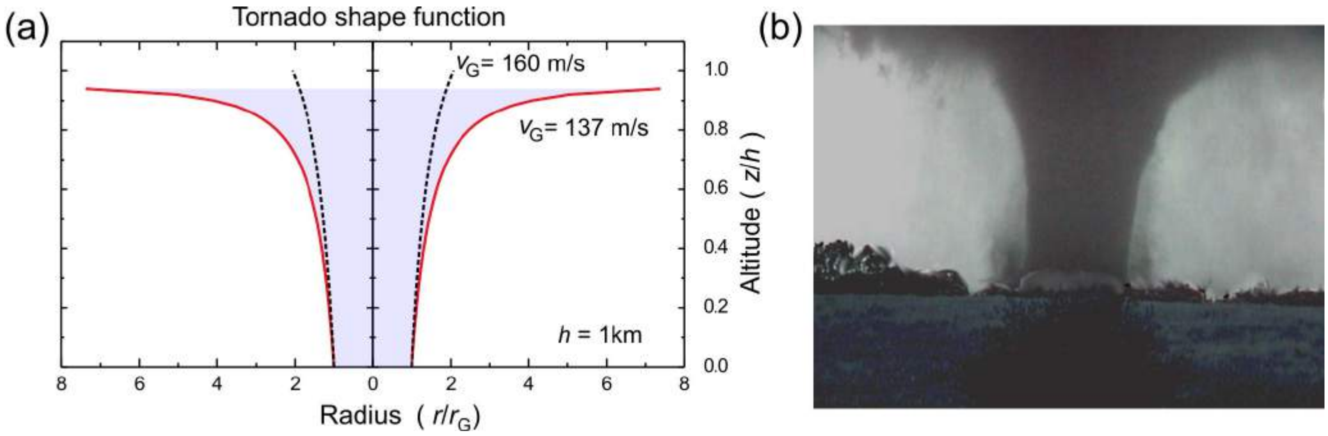


Figure 3. (a) The tornado shape function for two speeds with $h$ = 1 km. The higher the speed the more pipe-like it looks like. (b) A real life tornado for comparison (Valparaiso, Nebraska, US, 2001).

**(d)** Higher speed will make the term $2gz/v_G^2$ in Eq. (15) smaller, thus $r$ becomes a weaker function of $z$, in other words the tornado shape becomes more uniform or pipe-like. This is also illustrated in Figure 3(a).

## [Q.3] **The core of tornado**

**(a)** Consider a parcel of air rotating around the tornado as shown below:

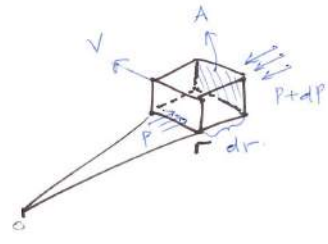


Figure 4. Parcel of air that rotates around the tornado

The differential pressure exerts the centrifugal force to the air parcel:

$$\Sigma F = m v^2 / r, \qquad \text{or:} \qquad dP\,A = \rho\,A\,dr\,v^2 / r \tag{16}$$

Thus we have:

$$\frac{\partial P}{\partial r} = \frac{\rho v^2}{r} \tag{17}$$

**(b)** Because the core is approximated as a rigid body, the angular velocity is constant everywhere inside the core, given as: $\omega = v_C / r_C$. Thus the velocity at any point inside the tornado:

$$v = \omega r = \frac{v_C\, r}{r_C} \tag{18}$$

So unlike in region I, in the core (region II) the velocity is proportional to the radius. The overall tangential velocity profile is given below:

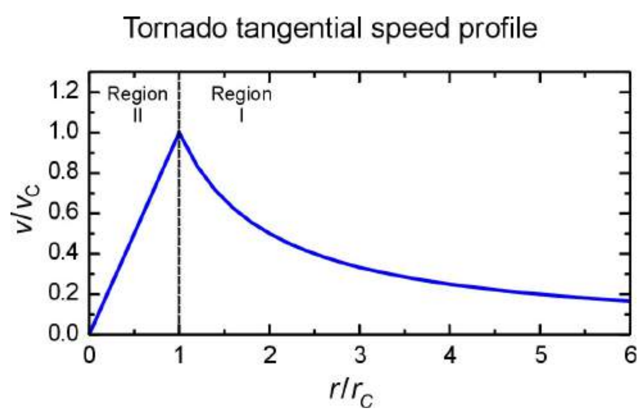


Figure 5. The tangential velocity profile of the tornado showing clear difference between region I and II.

Note that this model is in reasonably good qualitative agreement with an actual tornado measurement as shown below:

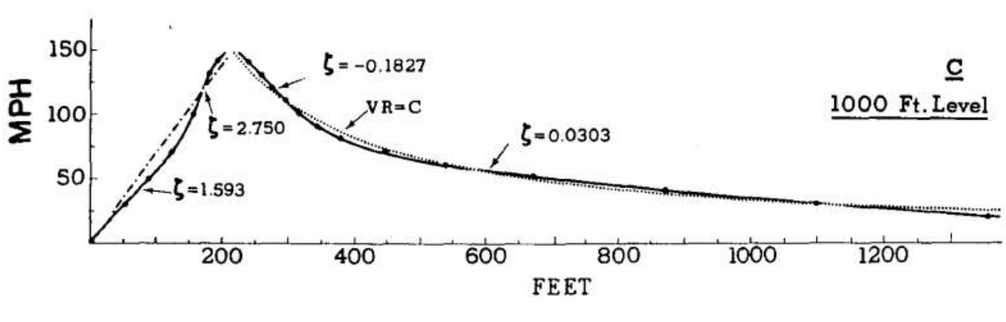


Figure 6. An actual tangential velocity profile of a tornado (Dallas, TX, 2 April 1957) [1].

**(c)** The pressure inside the tornado can be calculated with respect to a far away reference point on the ground where $P = P_0$ and using point G as a reference:

$$P_0 - P_T = \int_0^{\infty} \rho v^2 / r = \rho \int_0^{r_G} \frac{{v_G}^2 r}{{r_G}^2} dr + \rho \int_{r_G}^{\infty} \frac{{v_G}^2 {r_G}^2}{r^3} dr \tag{19}$$

$$P_0 - P_T = \tfrac{1}{2} \rho {v_G}^2 + \tfrac{1}{2} \rho {v_G}^2 {r_G}^2 \left. \frac{1}{r^2} \right|_{\infty}^{r_C} = \rho {v_G}^2 \tag{20}$$

The pressure inside tornado is given as:

$$P_T = P_0 - \rho {v_G}^2 \tag{21}$$

With the given data:

$$P_T = 1e5 - 1.2 \times 137^2 = 77{,}477 \,\text{Pa} \tag{22}$$

**(d)** Using ideal gas law and adiabatic assumption we have:

$$PV^{\gamma} = C_1, \text{ and } PV = nRT \tag{23}$$

$$P^{1-\gamma} T^{\gamma} = C_2 \tag{24}$$

where $\gamma$ is the heat capacity ratio which is for air : $\gamma = 1.4$.

Assuming a far away point with pressure $P_0$ and temperature $T_0$, the temperature at the core of tornado $T_T$ is:

$$T_T = T_0 \left( \frac{P_T}{P_0} \right)^{\frac{\gamma-1}{\gamma}} = 288 \times \left( \frac{77{,}477}{1e5} \right)^{\frac{0.4}{1.4}} = 268 \,\text{K} = \text{-}5\,^{\circ}\text{C} \tag{25}$$

Indeed it is very cold inside the tornado.

**(e)** Such a low temperature will cause condensation of moisture that get sucked into the tornado core, and the condensation will release ***latent heat***. Multiplied by massive influx of air mass to the tornado –this latent heat could provide the tremendous energy for the tornado.

## [Q.4] **Shall you open or close the windows?**

**(a)** We assume that the house is tightly enclosed maintaining the initial pressure at $P_0$. When the tornado approaches at a distance $d = 2\, r_C$ from the house, the differential pressure will be (Eq. 19):

$$\Delta P = P_0 - P_X = \rho \int_{2r_G}^{\infty} \frac{v_G^{\,2} r_G^{\,2}}{r^3} dr = \tfrac{1}{8} \rho v_G^{\,2} = \tfrac{1}{8} \times 1.2 \times 137^2 = 2{,}815\,\text{Pa} \tag{26}$$

The lift force on the roof (relative to weight)

$$\frac{F}{W} = \frac{\Delta P\, A}{A t\, \rho_{Roof}\, g} = \frac{\Delta P}{\rho_{Roof}\, g\, t} = \frac{2815}{800 \times 9.8 \times 0.1} = 3.6 \tag{27}$$

**(b)** The lift force is not much larger than the weight of the roof – and certainly most roof are mounted firmly to withstand forces multiple of its weight. So chances are the tornado pressure differential would not cause the house to explode so soon (unless the roof is very poorly mounted, or you are inside the tornado in which case this question becomes irrelevant).

Furthermore some residual opening in the house like ventilation and chimney will alleviate the pressure differential. However at a very close distance of 2 $r_{G.}$ it is more likely the house get smashed with flying debris. Thus you would just close all the windows.

# REFERENCES

[1] W.H. Hoecker, “Wind speed and air flow patterns in the Dallas tornado of April 2, 1957”, Mon. Wea. Rev. 88, 167-180.