WoPhO Selection Round Problem 5
The Rainbow

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1. A. We're going to denote the radius of the drop as R. In order to find f we calculate the total air resistant force acting on the drop: as the drop doesn't accelerate, this force must be equal to the gravitational force acting on the drop. Consider a small $\mathrm{d}A$ patch of the surface the radius drawn to which has an angle of α with the vertical dircetion. Then the force $f\mathrm{d}A$ acting on the patch has a vertical component $f\mathrm{d}A\sin\alpha$ (horizontal components of the force acting on symmetricly located patches cancel each other out).

Now instead of the patch let's imagine a horizontal spherical segment of height $\mathrm{d}h$ in a distance h from the equator (positive downwards). The surface of this segment is $\mathrm{d}A = 2\pi R\mathrm{d}h$, and since all patches of it have the same angle to the vertical, the vertical force acting on the segment can be calculated as $2\pi f R\mathrm{d}h\sin\alpha$. It is easy to see that the sine of α is $\frac{\sqrt{R^2-h^2}}{R}$, using this, the force element is $\mathrm{d}F = 2\pi f\sqrt{R^2-h^2}\cdot\mathrm{d}h$. The total air resistance is given by the integral of this over the domain of h: $F = \int_{h=-R}^{R}\mathrm{d}F = 2\pi f\int_{h=-R}^{R}\sqrt{R^2-h^2}\mathrm{d}h = 2\pi f\cdot\frac{\pi}{2}R^2 = \pi^2 f R^2$. This force must be equal with $mg = V\rho g = \frac{4}{3}\pi R^3\rho g$. Equating the two gives $f = \frac{4\rho g R}{3\pi} = \frac{2\rho g D}{3\pi}$ (we have used that the diameter is twice the radius).

B. In order to determine the tangential component of the drag force we're going to distract the given quarter in such segments as mentioned in part A. Moreover, every segments will be distracted into patches by vertical circles. The surface of one such patch will be $\mathrm{d}A = \frac{\mathrm{d}\beta}{2\pi}2\pi R\mathrm{d}h = R\mathrm{d}h\mathrm{d}\beta$, where $\mathrm{d}\beta$ is the dihedral angle of the spherical wedge used to cut out the patch: the angle β is measured from the central line of the quartersphere. Then the horizontal component of drag acting on the patch is $f\mathrm{d}A\cos\alpha$ (α is defined the same way as in part A); thus the component acting in the direction of the central line (the component perpendicular to this will be cancelled by patches at angles β and $-\beta$) is $\mathrm{d}F = f\mathrm{d}A\cos\alpha\sin\beta = Rf\cos\alpha\mathrm{d}h\sin\beta\mathrm{d}\beta$. We can express the cosine of α as $\frac{h}{R}$, thus $\mathrm{d}F = f\cdot h\mathrm{d}h\sin\beta\mathrm{d}\beta$. The total force is given by

$$F_a = \int_{h=0}^{R}\int_{\beta=-\pi/2}^{\pi/2}\mathrm{d}F = f\int_{h=0}^{R}h\mathrm{d}h\int_{\beta=-\pi/2}^{\pi/2}\sin\beta\mathrm{d}\beta = f\cdot\frac{R^2}{2}\cdot 2 = fR^2.$$

Substituting f found in part A gives $F_a = \frac{2\rho g D}{3\pi}\cdot\frac{D^2}{4} = \frac{\rho g D^3}{6\pi}$.

C. It is known that the pressure of curvature is given by $p = 2\sigma/R$. This pressure acts radially, thus the force acting in one specific direction on a given patch is $p\mathrm{d}A_n$ where $\mathrm{d}A_n$ is the surface of the perpendicular projection of the patch along the given direction. Summed up, the horizontal component of the surface tension is given by pA_n where A_n is the surface of the horizontal projection of the quartersphere: this is a semi-great-circle, the area of which is $\pi R^2/2$, thus the horizontal component of the surface tension is $2\sigma/R\cdot\pi/2R^2 = \pi\sigma R = \pi/2\sigma D$.

D. At the greatest spherical drop $F_t = 100F_a$: $\frac{\pi\sigma D_M}{2} = 100\frac{\rho g D_M^3}{6\pi}$; from this, $D_M = \sqrt{\frac{3}{100}\frac{\pi^2\sigma}{\rho g}} = 1.51$ mm.

2. A. Let β be the angle of refraction, one can see that the angle of income is α. It is known that the normal of a spherical surface is the radius. Let O be the centre of the sphere, A the point of income of the ray, B the point of the reflection and C the point where the ray goes out. Then angle OAB is β as stated; from the equilateral triangle OAB $OBA\angle$ will be β; from the laws of reflection then $OBC\angle = \beta$, so will $OCD\angle = \beta$, and due to symmetry, the angle of refraction at point C will be α. In the quadrilateral $OABC$ three angles are known (β, 2β, β); from this, it follows that the angle at O is $2\pi - 4\beta$; thus angle $COA = 4\beta$. One can see from this that the central angle of point C is $4\beta - \alpha$, this will be the angle of the normal at that point to the incident light. As the outgoing light has an angle α to this normal, the requested angle is $\theta = (4\beta - \alpha) - \alpha = 4\beta - 2\alpha$. Now we're going to calculate β: from Snell's law, $\sin\alpha/\sin\beta = n$, thus $\beta = \arcsin(\sin\alpha/n)$. Using this, $\theta = 4\arcsin(\sin\alpha/n) - 2\alpha$.

B. It is known that a small difference of a function can be calculated as $\Delta\theta = (\partial\theta/\partial\alpha)\Delta\alpha$. By calculating the derivative of the found function $\theta(\alpha)$ it follows that $\Delta\theta = \left(\frac{4\cos\alpha}{\sqrt{n^2-\sin^2\alpha}} - 2\right)\Delta\alpha$. Now we're calculating the area of the cross-section between the central angles α and $\alpha + \Delta\alpha$: this is a circular ring, the inner radius of which is $R\sin\alpha$, while the outer radius is $R\sin(\alpha+\Delta\alpha) = R(\sin\alpha + \cos\alpha\Delta\alpha)$: the area of the ring is $\Delta A = R^2((\sin\alpha + \cos\alpha\Delta\alpha)^2 - \sin^2\alpha) = 2R^2\sin\alpha\cos\alpha\Delta\alpha$. As the incoming intensity is I_0 all over the cross-section, the incoming power in this region is $\Delta P_0 = 2I_0R^2\sin\alpha\cos\alpha\Delta\alpha$. This power is lowered by transmittance and reflectance (denoted for a while as R_e) at surfaces, thus the outgoing power is $\Delta P = \Delta P_0 T_1 T_2 R_e = 2I_0 T_1 T_2 R_e R^2\sin\alpha\cos\alpha\Delta\alpha$.

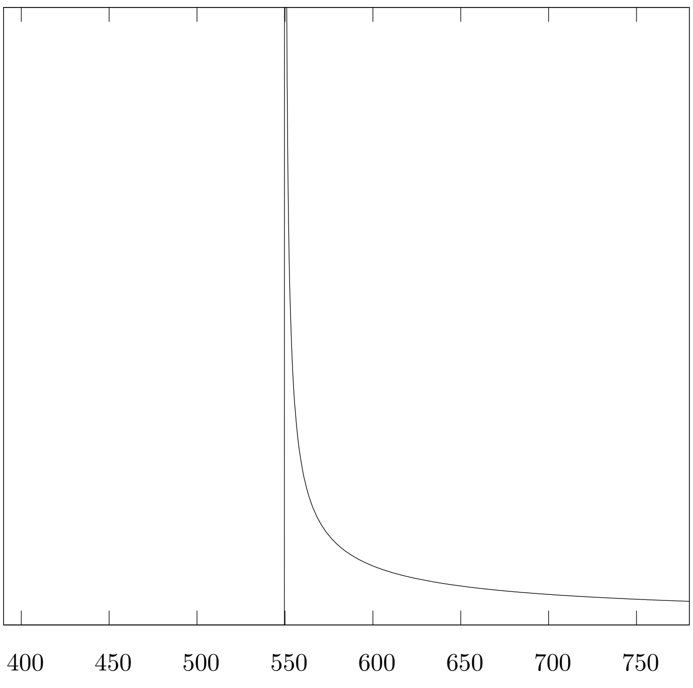
As the reflection of light is symmetrical in the coordinate ϕ, we can take the value of $\Delta\phi = 2\pi$ and ΔP the power reflected all along the ring at coordinate θ. Then the power distribution function is

$$J = \frac{\Delta P}{\Delta\theta \cdot 2\pi} = \frac{2I_0R^2 \sin\alpha\cos\alpha\Delta\alpha \cdot T_1T_2R_e}{2\pi \cdot \left(\frac{4\cos\alpha}{\sqrt{n^2-\sin^2\alpha}} - 2\right)\Delta\alpha} = \frac{I_0R^2T_1T_2R_e}{\pi}\frac{\sin\alpha\cos\alpha\sqrt{n^2-\sin^2\alpha}}{4\cos\alpha - \sqrt{n^2-\sin^2\alpha}} =$$

$$= \frac{I_0D^2T_1T_2R_e}{8\pi}\frac{\sin\alpha\cos\alpha\sqrt{n^2-\sin^2\alpha}}{2\cos\alpha - \sqrt{n^2-\sin^2\alpha}}.$$

C. We can see that there is an angle α where the denominator of the formula, $2\cos\alpha - \sqrt{n^2-\sin^2\alpha}$ is 0: at this point, J runs to infinity, thus the greatest part of the light will be reflected in a small environment of this angle. (Remark that over this angle, J is negative.) Solving the equation $2\cos\alpha - \sqrt{n^2-\sin^2\alpha} = 0$ gives, that $\sin\alpha = \sqrt{\frac{4-n^2}{3}}$. Substituting back in the formula for θ in part A: $\theta_M = 4\arcsin\left(\frac{4-n^2}{3n^2}\right) - 2\arcsin\left(\frac{4-n^2}{3}\right)$; at this angle, $J(\theta_M) \to \infty$. Evaluating θ_M for the given value of n_g gives $\theta_M = 41.9°$.

D. Let's plot a graph of $\theta(\alpha)$: it shows that the function has a maximum, that is, there will be no reflected light over the maximum of θ, what is exactly θ_M. The numerical value of θ_M is 40.5° for the violet light, 41.9° for green and 42.3° for red light. These mean that for the violet light there will be no reflected light along the direction $\theta = \theta_{M,g}$, but there will be reflected light in this direction for the red light. Consequently, as the critical θ grows monotonously with the wavelength, there will be some reflected intensity for $\lambda \geq 550$ nm. As stated before, at $\lambda = 550$ nm, the intensity runs to infinity, and as the intensity—wavelength function is continous, it will decrease monotonously. The sketch can be seen on Fig. 1: the only special point to be mentioned is the critical value of λ where the intensity jumps to infinity.



3. A. We follow the convention that the medium value of a dispersion-related quantity belongs to the wavelength of 550 nm. From this, the angular radius of the rainbow will be $\theta_0 = \theta_{M,g} = 41.9°$, because the (roughly) parallel coming rays will get distracted by this angle the most intensely. Because the sunrays come from a circle of angular radius $\delta/2$, all points that would be sharp points in the image of a parallel lightsource, will get blurred on a circle of diameter δ. Moreover, the rainbow would have an angular thickness $\theta_{M,r} - \theta_{M,v} = 1.76°$ because of the difference in the critical angle due to dispersion. These two effects will superpone, thus the angular width of the rainbow will be $1.76° + 0.5° = 2.3°$. Thus the angular radius of the rainbow is 41.9°, while its angular diameter is 2.3°.

B. The angular radius of the main diffraction circle made by an aperture of diameter d_m is $\theta = \arcsin\left(1.22\frac{\lambda}{d_m}\right)$; according to the problem text, this equals to the angular width of the rainbow: from this, $1.22\frac{\lambda}{d_m} = \sin\Delta\theta = 0.040$; one can see that the greater λ is, the greater will be the acceptable diameter d_m, consequently, d_m has to be calculated using the minimal λ of the visible spectrum, 390 nm. Then $d_m = 1.22\frac{\lambda_v}{0.040} = 11.9\ \mu$m.

C. When falling down, the drag force acting on the small, slowly falling raindrops is $6\pi\eta r v$ due to Stokes' law. This drag should equate the gravitational force on the drop, $mg = V\rho g = \frac{4}{3}\pi r^3 \rho g$. Equating the two force formulae gives the final velocity of the drop as $v = \frac{2}{9}\frac{\rho g r^2}{\eta} = \frac{1}{18}\frac{\rho g d^2}{\eta}$. As the drops are small, this final velocity is small, too, thus the drop will reach it fast. For the sake of simplicity, we assume that the final velocity is reached immediately. As the final velocity grows monotonously with the size of the drop, the smallest drops will fall the most slowly, thus they will be found the most time after the rain. The final velocity for the smallest drop that contributes to the rainbow is $v_m = \frac{1}{18}\frac{\rho g d_m^2}{\eta} = 4.3 \cdot 10^{-3}$ m/s; with this velocity it takes $T_M = \frac{s}{v_m} = 1.4 \cdot 10^5$ s to fall down the way $s = 800\text{ m} - 200\text{ m} = 600$ m.