WoPhO Selection Round Problem 7
Falling Ball Magnet

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1. A. It can be shown that such a ball magnet causes a magnetic field outside itself as if it were a single magnetic dipole of dipole moment $\mathbf{m} = \mathbf{M_0}V$, where V is the volume of the ball: $V = \frac{4}{3}\pi a^3$ (see for example: http://farside.ph.utexas.edu/teaching/jk1/lectures/node50.html). We shall use this fact in the following.

We're going to rephrase the problem: we're going to calculate the force required to move the tube constantly in the field of the fixed magnet with a velocity v: this is valid because of the equivalence of inertial systems (the system of the tube and the system of the magnet falling by v) and Newton's 3rd law, thus the force calculated this way is the same as the braking force exerted by the tube on the magnet.

In order to calculate this force, let's detach the tube in small rings of thickness $\mathrm{d}z$. In such a ring of distance z from the centre of the ball, the magnetic field induces an EMF of

$$U(z) = B_\perp(z)\ell v$$

due to Neumann's law, where ℓ is the length of the ring $\ell = 2a\pi$ and $B_\perp$ is the radial component of the field of the dipole (the field has no tangential component due to symmetry and Ampère's law; the tube-longitudinal component of the field doesn't induce any EMF since it's parallel to v). The resistance of the ring is

$$R = \rho\frac{\ell}{A} = \frac{1}{\sigma}\frac{\ell}{\Delta\cdot\mathrm{d}z}$$

where A is the area of the cross-section of the ring; the current flowing in this ring is then

$$\mathrm{d}I(z) = \frac{U(z)}{R} = \sigma\Delta B_\perp(z)v\mathrm{d}z;$$

the Joule power dissipated by this ring is then

$$\mathrm{d}P(z) = U(z)\mathrm{d}I(z) = \sigma\Delta B_\perp^2(z)v^2\cdot 2a\pi\mathrm{d}z.$$

The total power dissipated by the tube is then

$$P = \int_{z=-\infty}^{\infty}\mathrm{d}P(z) = \sigma\Delta v^2\cdot 2a\pi\int_{z=-\infty}^{\infty}B_\perp^2(z)\mathrm{d}z.$$

Here, we shall use that the magnetic field of a dipole can be expressed as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(-\frac{\mathbf{m}}{r^3} + \frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5}\right),$$

the radial component of which in a point of axial distance z and radial distance a from the dipole is

$$B_\perp(z) = \frac{\mu_0}{4\pi}\frac{3m\cdot za}{r^5} = \frac{3\mu_0 m}{4\pi}\frac{za}{(z^2+a^2)^{5/2}} = \frac{3\mu_0}{4\pi}\frac{4\pi}{3}a^3M_0\frac{za}{(z^2+a^2)^{5/2}} = \mu_0 M_0 a^4\cdot\frac{z}{(z^2+a^2)^{5/2}};$$

substituting this into the expression of P gives

$$P = \sigma\Delta v^2\cdot 2a\pi\cdot\mu_0^2M_0^2a^8\int_{z=-\infty}^{\infty}\frac{z^2\mathrm{d}z}{(z^2+a^2)^5} = 2\pi\mu_0^2\sigma\Delta M_0^2a^9v^2\int_{\alpha=-\infty}^{\infty}\frac{a^2\alpha^2\cdot a\mathrm{d}\alpha}{a^{10}(\alpha^2+1)^5} =$$

$$= 2\pi\mu_0^2\sigma\Delta M_0^2a^2v^2\int_{\alpha=-\infty}^{\infty}\frac{\alpha^2\mathrm{d}\alpha}{(1+\alpha^2)^5}.$$

(We have used the substitution $z = \alpha a$ in order to make the integral dimensionless.) It can be found (by using mathematical programs or by looking up the antiderivative in an integral table) that the improper integral exists and its value is $\frac{5\pi}{128}$; consequently, the value of the power is

$$P = 2\pi\mu_0^2\sigma\Delta M_0^2a^2v^2\cdot\frac{5\pi}{128} = \frac{5\pi^2}{64}\mu_0^2\sigma\Delta M_0^2a^2v^2.$$

This power is dissipated by the tube: if we want to maintain the velocity, this power should be entered into the system mechanically. It is well-known, that the mechanical power can be calculated as $P = Fv$ (obviously, the force must be parallel to the velocity), consequently the required force is

$$F = \frac{P}{v} = \frac{5\pi^2}{64}\mu_0^2\sigma\Delta M_0^2 a^2 v.$$

According to the Newton equation, this force brakes the tube with respect to the magnet (the total force acting on a uniformly moving body must be zero); and due to Newton's 3rd law, this force brakes the magnet with respect to the tube. The magnetic braking force is:

$$F = \frac{5\pi^2}{64}\mu_0^2\sigma\Delta M_0^2 a^2 v.$$

B. When the magnet reaches its terminal velocity, there acts no total force on it, that is, the braking force is equal to the gravitational force:

$$mg = \frac{5\pi^2}{64}\mu_0^2\sigma\Delta M_0^2 a^2 v_t$$

$$v_t = \frac{mg}{\frac{5\pi^2}{64}\mu_0^2\sigma\Delta M_0^2 a^2} = \frac{64}{5\pi^2}\frac{mg}{\mu_0^2\sigma\Delta M_0^2 a^2}.$$

C. The slope of the line defining τ is the value of $\frac{\mathrm{d}v}{\mathrm{d}t}$ at $t = 0$, that is, the initial acceleration of the magnet. Since there acts no magnetic braking force at $v = 0$, the only force is the gravitational force mg, thus the inital acceleration will be g: from the right-angled triangle that can be seen in the figure, the time scale is given as

$$\tau = \frac{v_t}{g} = \frac{64}{5\pi^2}\frac{m}{\mu_0^2\sigma\Delta M_0^2 a^2}.$$

Remark: the equation of the $v(t)$ curve.
The total force exerted on the magnet is given by

$$F_t = mg - \frac{5\pi^2}{64}\mu_0^2\sigma\Delta M_0^2 a^2 v = \frac{5\pi^2}{64}\mu_0^2\sigma\Delta M_0^2 a^2 (v_t - v),$$

thus its acceleration will be

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{F_t}{m} = \frac{5\pi^2}{64}\frac{\mu_0^2\sigma\Delta M_0^2 a^2}{m}(v_t - v) = \frac{v_t - v}{\tau},$$

using the parameters v_t and τ calculated before. The solution of this separable differential equation corresponding to the initial condition $v(0) = 0$ is

$$v(t) = v_t(1 - e^{-t/\tau}).$$

This is the function of $v(t)$.