WoPhO Selection Round Problem 8
Relativistic Images

Attila Szabó, Grade 11
Leőwey Klára High School
Pécs, Hungary

1. A. In this part we take $c = 1$ for simplicity. Let's consider the process in the inertial frame of the mirror. Let the impulse of the incident photon be (p_x, p_y) in this frame (the mirror moves in the $-x$ direction, $+y$ direction points „upwards"). Then the impulse of the photon reflected from the standing mirror is $(-p_x, p_y)$ as the energy of the photon doesn't change and the angle of incidence is the same as the angle of reflection (this is well-known for standing mirrors). The energy of both photons is $W = \sqrt{p_x^2 + p_y^2}$. Now we go back to the rest frame: as the mirror moves with v to $-x$ in this frame, the velocity of this frame is v to the $+x$ direction compared to the mirror frame. We can use the Lorentz transformation on the energy–impulse four-vector to get the parameters of the incident and the reflected photon in the rest frame. Due to the Lorentz transformation, the y-components of the incident and the reflected photons' impulse are p_y; the energy of the photons can be calculated as:

$$W_i = \frac{W - p_x v}{\sqrt{1 - v^2}}; \quad W_r = \frac{W - (-p_x)v}{\sqrt{1 - v^2}} = \frac{W + p_x v}{\sqrt{1 - v^2}}.$$

Since the energy and the magnitude of the impulse of a photon is the same (as $c = 1$), $\sin\alpha = \frac{p_y}{W_i}$ and $\sin\beta = \frac{p_y}{W_r}$. Substituting the found values of energies and $\tan\gamma = \frac{p_y}{p_x}$, $\sin\delta = v$ gives

$$\sin\alpha = \frac{\sin\gamma\cos\delta}{1 - \cos\gamma\sin\delta}; \quad \sin\beta = \frac{\sin\gamma\cos\delta}{1 + \cos\gamma\sin\delta}.$$

Expressing $\sin\beta$ using α and δ (via mathematical programs e.g. Maple) gives

$$\sin\beta = \frac{\sin\alpha\cos^2\delta}{2(1 + \cos\alpha\sin\delta) - \cos^2\delta} = \frac{\sin\alpha\cos^2\delta}{1 + \sin^2\delta + 2\sin\delta\cos\alpha} = \frac{\sin\alpha(c^2 - v^2)}{c^2 + 2vc\cos\alpha + v^2};$$

we have used that $\sin\delta = v$, and substituted back the ignored terms of c. If $\alpha \to 0$ then $\cos\alpha = 1$, thus

$$\sin\beta = \sin\alpha\frac{c^2 - v^2}{c^2 + 2vc + v^2};$$

as for small angles $\sin x = x$ holds, the following comes:

$$\frac{\alpha}{\beta} = \frac{\sin\alpha}{\sin\beta} = \frac{c^2 + 2vc + v^2}{c^2 - v^2} = \frac{(c + v)^2}{(c + v)(c - v)} = \frac{c + v}{c - v};$$

that was to be proven.

B. (a) Let's examine the situation from the rest frame of the camera. Due to Lorentz contraction, the distance of the object to the camera is $L' = L\sqrt{1 - \frac{v^2}{c^2}}$, whilst its height does not change: $H' = H$. However, the camera shows the image of the object that was in a distance L_1 from it: since this point, the distance covered by the light ct, while that of the object is vt: their difference is $ct - vt = L'$, thus $t = \frac{L'}{c-v}$, so $L_1 = ct = \frac{L'}{1-v/c} = L\sqrt{\frac{1+v/c}{1-v/c}} = L\sqrt{\frac{c+v}{c-v}}$. This is the distance of the object to the camera that can be seen at the place of the camera: the image will be formed as if the object were in a distance of L_1. Using the formula for image height gives the height as $h' = \frac{H'f}{L_1} = \frac{Hf}{L}\sqrt{\frac{c-v}{c+v}} = h\sqrt{\frac{c-v}{c+v}}$. Thus the image will be smaller compared to the camera in rest.

(b) The rate of variation is $\dot{h'}$. We're going to calculate this rate in the camera frame: due to Lorentz dilatation, the eigentimespan can be calculated from the inertial timespan as $\mathrm{d}t' = \mathrm{d}t\sqrt{1 - \frac{v^2}{c^2}}$, thus

$$\dot{h'} = Hf\sqrt{\frac{c - v}{c + v}}\left(\frac{1}{L}\right)^{\cdot} = Hf\sqrt{\frac{c - v}{c + v}}\frac{-\dot{L}}{L^2} = \frac{Hf}{L^2}\sqrt{\frac{c - v}{c + v}}\frac{v\mathrm{d}t}{\mathrm{d}t\sqrt{1 - \frac{v^2}{c^2}}} = \frac{Hfv}{L^2}\frac{1}{1 + v/c}.$$

(During an inertial timespan $\mathrm{d}t$, the object and the camera get closer by $v\mathrm{d}t$.) Thus the image height will grow with a rate of $\frac{Hfv}{L^2}\frac{1}{1+v/c}$ as seen from the camera reference frame.

C. (a) We examine the process from the camera rest frame. The camera–mirror distance in this frame is $L_1 = L\sqrt{1-\frac{v^2}{c^2}}$. As the mirror approaches the camera with v, the distance between the camera and the mirror was $L_2 = \frac{L_1}{1-v/c} = L\sqrt{\frac{c+v}{c-v}}$ when the light recorded at the examined moment was reflected (to conclude this, we can use the same ideas as in 1. B. (a)).

Now we determine the parameters of the virtual image at this point. Since the perpendicularly incoming rays are reflected perpendicularly, the image points will be at the same vertical height as the respective points of the camera: this causes that the width of the camera and its mirror image is the same. Now check a light ray that comes from a point of the camera to the mirror covering a distance L_2 horizontally and $x \ll L_2$ vertically: if the distance of the image is L_3, the tangent of the angle of incidence and reflection is $\tan\alpha = \frac{x}{L_2}$, $\tan\beta = \frac{x}{L_3}$; since x is small, $\tan\alpha = \alpha$ and $\tan\beta = \beta$ can be used. Using the small angle limit for the relativistic mirror gives $\frac{c+v}{c-v} = \frac{\alpha}{\beta} = \frac{\tan\alpha}{\tan\beta} = \frac{x/L_2}{x/L_3} = \frac{L_3}{L_2}$, thus $L_3 = L_2\frac{c+v}{c-v}$. The size of the image is D while its distance from the mirror is L_3.

Consequently, the camera records an image of width D in an effective distance $L' = L_2 + L_3 = L_2\left(1+\frac{c+v}{c-v}\right) = L_2\frac{2c}{c-v} = L\sqrt{\frac{c+v}{c-v}}\frac{2c}{c-v}$. Using the formula of image size given by the problem gives $d = \frac{Df}{L'} = \frac{Df}{L}\sqrt{\frac{c-v}{c+v}}\frac{c-v}{2c}$.

We're going to calculate the rate of change in the camera frame (since the video will show this). A very similar calculation as in part 1. B. (b) gives

$$\dot{d} = Df\sqrt{\frac{c-v}{c+v}}\frac{c-v}{2c}\frac{-\dot{L}}{L^2} = \frac{Df}{L^2}\sqrt{\frac{c-v}{c+v}}\frac{c-v}{2c}\frac{v\mathrm{d}t}{\mathrm{d}t\sqrt{1-\frac{v^2}{c^2}}} = \frac{Df}{L^2}\frac{v}{2c}\frac{c-v}{c+v}.$$

This is the rate of change of the image width.

(b) The difference between the reading of the clocks is the time required by the light to cover the distance to the mirror and back:

$$A = \frac{2L_2}{c} = \frac{2L}{c}\sqrt{\frac{c+v}{c-v}};$$

obviously the image of the clock shows an earlier time. Consequently, the reading t' of the image and that t of the clock is connected by $t = t' + \frac{2L}{c}\sqrt{\frac{c+v}{c-v}}$. Taking differentials: $\mathrm{d}t = \mathrm{d}t' + \frac{2}{c}\mathrm{d}L\sqrt{\frac{c+v}{c-v}}$. As the distance in the camera frame is given by $L_1 = L\sqrt{1-\frac{v^2}{c^2}}$, its differential is $\mathrm{d}L_1 = \mathrm{d}L\sqrt{1-\frac{v^2}{c^2}}$; on the other hand, this is $\mathrm{d}L_1 = -v\mathrm{d}t$ due to the definition of the velocity of the mirror in the camera frame: consequently, $\mathrm{d}L = -\frac{v\mathrm{d}t}{\sqrt{1-\frac{v^2}{c^2}}}$. Substituting this into the previous equation:

$$\mathrm{d}t = \mathrm{d}t' - 2v\sqrt{\frac{c+v}{(c-v)(c^2-v^2)}}\mathrm{d}t = \mathrm{d}t' - \frac{2v}{c-v}\mathrm{d}t$$

$$\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{c+v}{c-v};$$

thus during each tick of the clock one can observe $\frac{c+v}{c-v}$ ticks of the image, that is,

$$B = \frac{c+v}{c-v}.$$

The task is to determine L from A and B: it is easy to see that $L = \frac{A}{\sqrt{B}}\frac{c}{2}$. This was to be found.

(c) If the camera reaches the mirror, L comes to be 0, the difference between the readings will be $A = \frac{2L}{c}\sqrt{\frac{c+v}{c-v}} = 0$, that is, both clocks will show the same time. This was to be proven.

2. A. We're going to determine the apparent angular radius t time after the „explosion" (we define $\theta(t)$ this way). In order to do so, let's check a light ray that reaches the observer at an angle α measured from the Earth–star central line. Let's assume it has travelled τ from the star to the observer; consequently, it started from the surface of the star star $t-\tau$ after the explosion; at this time, the radius of the star was $R_0 + v(t-\tau)$. We can use the law of cosines for the triangle formed by the observer, the centre of the star and the light source at the vertex of the observer:

$$(R_0 + v(t-\tau))^2 = L^2 + c^2\tau^2 - 2Lc\tau\cos\alpha$$

$$(c^2 - v^2)\tau^2 + 2(v^2 t + R_0 v - Lc\cos\alpha)\tau + (L^2 - (R_0 + vt)^2) = 0.$$

The number of real roots of this quadratic equation depends on the sign of its discriminant: if $D \geq 0$ there is at least one solution for τ (therefore, at the specific angle α there will be light); however, if $D < 0$, there will be no corresponding light source: at the boundary of the star, $D = 0$ will hold:

$$D = 4(v^2 t + R_0 v - Lc\cos\theta)^2 - 4(c^2 - v^2)(L^2 - (R_0 + vt)^2) = 0$$

$$v(R_0 + vt) - Lc\cos\theta = \pm\sqrt{c^2 - v^2}\sqrt{L^2 - (R_0 + vt)^2}$$

$$\cos\theta = \frac{v}{c}\frac{R_0 + vt}{L} \pm \sqrt{1 - \frac{v^2}{c^2}}\sqrt{1 - \frac{(R_0 + vt)^2}{L^2}}$$

Let α and β such acute angles that $\cos\alpha = \frac{v}{c}$ and $\cos\beta = \frac{R_0 + vt}{L}$. It is easy to see that $\cos\theta$ must be positive, therefore we retain the positive root: using the given substitution, it will be $\cos(\alpha - \beta)$, thus

$$\theta = \alpha - \beta = \arccos\left(\frac{v}{c}\right) - \arccos\left(\frac{R_0 + vt}{L}\right).$$

Now we take into account that the observer notes the phenomenon at $t_0 = \frac{L - R_0}{c}$ due to the speed of light; that is, it's more natural to measure time on a time scale $t' = t - t_0$: using this substitution, we get

$$\theta = \arccos\left(\frac{v}{c}\right) - \arccos\left(\frac{R_0 + vt' + \frac{v}{c}L - \frac{v}{c}R_0}{L}\right) = \arccos\left(\frac{v}{c}\right) - \arccos\left(\frac{v}{c} + \frac{\frac{c-v}{c}R_0 + vt'}{L}\right).$$

This way depends the angular size of the star on time.

Remark. If t' is small, then $\frac{\frac{c-v}{c}R_0 + vt'}{L}$ is small, too, so we can approximate the formula of θ to the first order in it:

$$\theta = -\frac{\frac{c-v}{c}R_0 + vt'}{L}\cdot\arccos'\left(\frac{v}{c}\right) = \frac{1}{\sqrt{1 - v^2/c^2}}\frac{\frac{c-v}{c}R_0 + vt'}{L}.$$

We should note, that at $t' = 0$ this is definitely smaller than $\arcsin\left(\frac{R_0}{L}\right)$, the apparent radius of the star before the explosion: this problem is caused by that during calculations we haven't checked whether $t - \tau > 0$, in other words, whether the star has exploded already. At small values of t' it's possible that we took into account such extrapolated states of the star in which its radius is smaller than R_0 that doesn't happen actually. This problem can be fixed the most easily this way: the radius of the star is

$$\theta = \max\left[\arcsin\left(\frac{R_0}{L}\right); \arccos\left(\frac{v}{c}\right) - \arccos\left(\frac{v}{c} + \frac{\frac{c-v}{c}R_0 + vt'}{L}\right)\right]:$$

this formula discards effects of extrapolation explicitly.

B. Since the system is rotationally symmetric about the Earth–star central line, the image of the star must be symmetric, too: this means, that the image will be formed of circular ring(s), the task is to determine its radius/their radii.

Now we determine when (at which time instant t') will the observer see the blackout at an angle α to the centre of the star. Then the observer will obviously see the star when its radius was R_m. If the distance that the light had to cover was x then we can write down the law of cosines for the observer vertex of the triangle observer–centre of star–light (or darkness) source:

$$R_m^2 = L^2 + x^2 - 2Lx\cos\alpha$$

$$x^2 - 2Lx\cos\alpha + (L^2 - R_m^2) = 0$$

The smaller root of this equation in x (the larger root corresponds to the other side of the star, the light from that never reaches the observer) is

$$x = L\cos\alpha - \sqrt{R_m^2 - L^2\sin^2\alpha}$$

The time required by the star to expand to R_m is $t_1 = \frac{R_m - R_0}{v}$, while the time required by the light to reach the observer is $t_2 = \frac{x}{c}$; the time when the observer sees the blackout is $t = t_1 + t_2$, or in the epoch t' introduced before:

$$t' = t_1 + t_2 - t_0 = R_0 \frac{v-c}{vc} + \frac{R_m}{v} + \frac{x-L}{c} = R_0 \frac{v-c}{vc} + \frac{R_m}{v} - \frac{L(1-\cos\alpha) + \sqrt{R_m^2 - L^2 \sin^2\alpha}}{c}$$

The plot of both $\theta(t')$ both $\alpha(t')$ for some R_0, R_m and L can be seen in the figure. One can see that the plot of $\alpha(t)$ is tangent to that of $\theta(t)$.

The plot means the following: at $t' \approx 5$ s the edge of the star starts to expand and the apparent radius starts to grow. At $t' \approx 135$ s, there appears a dark spot in the middle of the star (this is the first sign of the blackout of the star) that starts to grow and at $t' \approx 170$ s it will become as large as the star itself: this means, that after this point teh whole star will appear dark. The apparent shape of the star is a bright circular ring, the apparent radii of which are $\alpha(t')$ and $\theta(t')$ that can be determined using the derived formulae: at some point in time, $\alpha(t') = \theta(t')$ will occur: after this point, the star will look like a dark circle of radius $\theta(t')$.