WoPhO Selection Round Problem 3
Particles in the magnetosphere
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Part 1. Path around field lines

(a) The unit magnetic field is $B_E = \frac{\mu_0 M_E}{4\pi R_E^3}\sqrt{(2\cdot\sin 0)^2+\cos^2 0} = \frac{\mu_0 M_E}{4\pi R_E^3}$. Using this unit, the modulus of the magnetic field is (noting that $\mathbf{u}_r \perp \mathbf{u}_\lambda$ thus we can use the Pythagorean theorem for adding the components:

$$B(L,\lambda) = \frac{\mu_0 M_E}{4\pi r^3}\sqrt{(2\sin\lambda)^2+\cos^2\lambda} = \frac{\mu_0 M_E}{4\pi R_{\text{Eq}}^3\cos^6\lambda}\sqrt{(2\sin\lambda)^2+\cos^2\lambda} = \frac{\mu_0 M_E}{4\pi R_E^3}\frac{\sqrt{1+3\sin^2\lambda}}{L^3\cos^6\lambda}$$

$$B(L,\lambda) = B_E\frac{\sqrt{1+3\sin^2\lambda}}{L^3\cos^6\lambda}.$$

(b) The Lorentz force acting on the particle is $qv_\perp B$: this force should hold the particle on the circular orbit by maintaining the centripetal acceleration $v_\perp\omega_g$. Newton's equation in this case: $qv_\perp B = mv_\perp\omega_g$, from this,

$$\omega_g = \frac{qB}{m} = \frac{eB_E}{m_p}\frac{\sqrt{1+3\sin^2\lambda}}{L^3\cos^6\lambda}.$$

As $\tan\alpha = v_\perp/v_\parallel$, $v/v_\parallel = \sqrt{v_\parallel^2+v_\perp^2}/v_\parallel = \sqrt{1+\tan^2\alpha} = 1/\cos\alpha$, thus $v_\parallel = v\cos\alpha$ and $v_\perp = v_\parallel\tan\alpha = v\sin\alpha$. By the definition of kinetic energy and angular velocity, $v = \sqrt{2W/m_p}$ and $v_\perp = r_g\omega_g$, thus

$$r_g = \frac{v_\perp}{\omega_g} = \frac{v\sin\alpha}{\omega_g} = \frac{\sqrt{2W/m_p}\sin\alpha}{eB_E/m_p}\frac{L^3\cos^6\lambda}{\sqrt{1+3\sin^2\lambda}} = \frac{\sqrt{2Wm_p}\sin\alpha}{eB_E}\frac{L^3\cos^6\lambda}{\sqrt{1+3\sin^2\lambda}}.$$

Part 2. Mirror points

(a) The gyrating proton will resemble a current $I = q/T = \frac{e}{2\pi/\omega_g} = e\omega_g/2\pi$ enclosing an area of $A = \pi r_g^2$. The magnetic moment of such a current is

$$\mu = IA = \frac{e\omega_g}{2\pi}\cdot\pi r_g^2 = \frac{e}{2}r_g^2\omega_g = \frac{e}{2}\frac{2Wm_p\sin^2\alpha}{e^2B_E^2}\left(\frac{L^3\cos^6\lambda}{\sqrt{1+3\sin^2\lambda}}\right)^2\cdot\frac{eB_E}{m_p}\frac{\sqrt{1+3\sin^2\lambda}}{L^3\cos^6\lambda}$$

$$\mu = \frac{W\sin^2\alpha}{B_E}\frac{L^3\cos^6\lambda}{\sqrt{1+3\sin^2\lambda}}.$$

(b) At the Equator, $\lambda = 0$ and $\alpha = \alpha_{\text{Eq}}$, thus $\mu = WL^3\sin^2\alpha_{\text{Eq}}/B_E$. At the mirror point, $v_\parallel = 0$, thus $\alpha = 90°$ and $\lambda = \lambda_m$; in this case, $\mu = WL^3/B_E\cdot\cos^6\lambda_m/\sqrt{1+3\sin^2\lambda_m}$. As μ and W doesn't vary (magnetic field doesn't do any work on charges),

$$\sin^2\alpha_{\text{Eq}} = \frac{\cos^6\lambda_m}{\sqrt{1+3\sin^2\lambda_m}}.$$

If $\alpha_{\text{Eq}} = 30°$, the numeric solution of the above equation gives $\lambda_m = 33.15°$.

Part 3. Particle collision

(a) The collision takes place if $r_{\min} \le R_E$; however, $r_{\min} = r_{\text{Eq}}\cos^2\lambda_m = LR_E\cos^2\lambda_m$, thus the condition is $\cos^2\lambda_m \le 1/L$. From Part 2b we have

$$\sin^2\alpha_{\text{Eq}} = \frac{\cos^6\lambda_m}{\sqrt{4-3\cos^2\lambda_m}} \le \frac{(1/L)^3}{\sqrt{4-3/L}}$$

$$\alpha_{\text{Eq}} \le \arcsin\left(\frac{1}{L^{3/2}\sqrt[4]{4-3/L}}\right).$$

This is the *maximal* equatorial pitch angle needed for the collision to take place.
(There cannot be any minimal such angle. If this were the case, a proton with a pitch angle of 90° would collide

with Earth as well: this is impossible, since the proton wouldn't have any field-directed velocity to move towards the Earth. In general, the pitch angle grows with the magnetic field getting stronger.)

(b) We're going to express the time required to change λ by $\mathrm{d}\lambda$ in order to integrate it. The velocity component of the proton parallel with the field line is $v\cos\alpha = \sqrt{2W/m_p}\,\cos\alpha$. Therefore we need to express α from the conservation of μ: by simplifying with the common terms WL^3/B_E it gives

$$\sin^2\alpha_{\mathrm{Eq}} = \sin^2\alpha\frac{\cos^6\lambda}{\sqrt{1+3\sin^2\lambda}} \to \sin^2\alpha = \sin^2\alpha_{\mathrm{Eq}}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda} \to \cos\alpha = \sqrt{1-\sin^2\alpha_{\mathrm{Eq}}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}}.$$

The distance to be travelled along the field line is the line element in the polar coordinates specified by the problem:

$$\mathrm{d}s = \sqrt{(\mathrm{d}r)^2 + r^2(\mathrm{d}\lambda)^2} = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda^2}\right)}\mathrm{d}\lambda = \sqrt{r_{\mathrm{Eq}}^2\cos\lambda^4 + 4r_{\mathrm{Eq}}^2\sin^2\lambda\cos^2\lambda}\;\mathrm{d}\lambda$$

$$\mathrm{d}s = LR_E\cos\lambda\sqrt{1+3\sin^2\lambda}\;\mathrm{d}\lambda.$$

The time element to cover this distance is consequently

$$\mathrm{d}t = \frac{\mathrm{d}s}{v} = \frac{LR_E}{\sqrt{2W/m_p}}\frac{\cos\lambda\sqrt{1+3\sin^2\lambda}}{\sqrt{1-\sin^2\alpha_{\mathrm{Eq}}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}}}\mathrm{d}\lambda.$$

We need to integrate this time element along the cycle: due to symmetry it splits into four identical sections, one being between from $\lambda = 0$ to $\lambda = \lambda_m$ ($LR_E\cos^2\lambda_m > R_E$ in our case, knowing the value of λ_m from Part 2b, thus the proton turns back at the mirror point rather than at the surface of the Earth). Using this, the period is

$$T = \frac{4LR_E}{\sqrt{2W/m_p}}\int_{\lambda=0}^{\lambda_m}\frac{\cos\lambda\sqrt{1+3\sin^2\lambda}}{\sqrt{1-\sin^2\alpha_{\mathrm{Eq}}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}}}\mathrm{d}\lambda.$$

The integral can only be evaluated numerically; its value according to GNU Octave (scripts for quadrature attached) is 0.999729 rounded to 6 decimal places. Substituting this and the numeric values of variables into the expression of T gives $T = 2.924$ s.

(c) As the kinetic energy is constant all along the path (magnetic field does no work on the proton), the velocity is constant as well. The total distance covered by the proton, i.e. the length of the path is simply this velocity times the period:

$$\ell = T\sqrt{\frac{2W}{m_p}} = 4LR_E\int_{\lambda=0}^{\lambda_m}\frac{\cos\lambda\sqrt{1+3\sin^2\lambda}}{\sqrt{1-\sin^2\alpha_{\mathrm{Eq}}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}}}\mathrm{d}\lambda;$$

its numeric value is $\ell = 1.28\cdot 10^8$ m.

Attached files

`integrand.m`, `integrator.m`: the GNU Octave scripts that describe the integrand and do the integration; setting the working directory of Octave into their directory and calling `integrator` gives the value of the requested integral.